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Sheet 01: Mathematical Foundations

Posted: Mo 13.10.25 Central Tutorial: Th 16.10.27 Due: Th.24.10.25, 14:00

(b)[2](E/M/A) means: problem (b) counts 2 points and is easy/medium hard/advanced

Suggestions for central tutorial: example problems 9, 10, 4, 3.

Videos exist for example problems 9 (C2.3.1), 10 (C2.3.3).

Example Problem 1: Composition of maps [2]

Points: (a)[1](E); (b)[1](E).

Let $\mathbb{N}_0$ denote the set of all natural numbers including zero, and $\mathbb{Z}$ the set of all integers. Consider the following two maps:

$$\begin{aligned} A : \mathbb{Z} &\rightarrow \mathbb{Z}, & n &\mapsto A(n) = n + 1, \\ B : \mathbb{Z} &\rightarrow \mathbb{N}_0, & n &\mapsto B(n) = |n| \equiv n \cdot \text{sign}(n). \end{aligned}$$

(a) Find the composite map $C = B \circ A$, i.e. specify its domain, image and action on n .

(b) Which of the above maps A , B and C are surjective? Injective? Bijective?

Example Problem 2: The abelian group $\mathbb{Z}_2$ [3]

Points: (a)[2](E); (b)[1](E).

(a) Show that $\mathbb{Z}_2 \equiv (\{0, 1\}, +)$, where the addition operation $+$ is defined by the adjacent composition table, is an abelian group.

+	0	1
0	0	1
1	1	0

(b) Construct a group isomorphic to $\mathbb{Z}_2$, using two integers as group elements and standard multiplication of integers as group operation. Set up the corresponding composition table.

Example Problem 3: Permutation groups [4]

Points: (a)[3](E); (b)[0,5](E); (c)[0,5](E).

A map which reorders n labelled objects is called a **permutation** of these objects. For example, $1234 \xrightarrow{[4312]} 4312$ is a permutation of the four numbers in the string 1234, where we use $[4312]$ as shorthand for the map $1 \mapsto 4, 2 \mapsto 3, 3 \mapsto 1$ and $4 \mapsto 2$. Similarly, if the same permutation is applied to the string 2314, it yields $2314 \xrightarrow{[4312]} 3142$. (In general, $[P(1) \dots P(n)]$ denotes the map $j \mapsto P(j)$ which replaces j by $P(j)$, for $j = 1, \dots, n$.) Two permutations performed in succession again yield a permutation. For example, acting on 1234 with $P = [4312]$ followed by $P' = [2413]$ yields $1234 \xrightarrow{[4312]} 4312 \xrightarrow{[2413]} 3124$, thus the resulting permutation is $P' \circ P = [3124]$.

The set of all possible permutations of n numbers, denoted by S_n , contains $n!$ elements. Viewing $P' \circ P$ (perform first P , then P') as a group operation,

$$\circ : S_n \times S_n \rightarrow S_n, \quad (P', P) \mapsto P' \circ P,$$

we obtain a group, $(S_n, \circ)$, the **permutation group**, usually denoted simply by S_n .

- (a) Complete the adjacent composition table for S_3 , in which the entries $P' \circ P$ are arranged such that those with fixed P' sit in the same row, those with fixed P in the same column.

$P' \circ P$	[123]	[231]	[312]	[213]	[321]	[132]
[123]	[123]	[231]	[312]	[213]	[321]	[132]
[231]		[312]	[123]	[321]	[132]	[213]
[312]			[231]	[132]	[213]	[321]
[213]					[312]	[231]
[321]						[312]
[132]						

- (b) Which element is the neutral element of S_3 ? How can we see from the multiplication table that every element has a unique inverse?
- (c) Is S_3 an abelian group? Justify your answer.

Example Problem 4: Algebraic manipulations with complex numbers [4]

Points: (a-c)[0,5](E); (d)[0,5](M); (e)[0,5](E); (f)[0,5](E); (g)[1](M); (h)[1](M).

For $z = x + iy \in \mathbb{C}$, bring each of the following expressions into standard form, i.e. write them as (real part) + i(imaginary part):

- (a) $z + \bar{z}$, (b) $z - \bar{z}$, (c) $z \cdot \bar{z}$, (d) $\frac{z}{\bar{z}}$,
 (e) $\frac{1}{z} + \frac{1}{\bar{z}}$, (f) $\frac{1}{z} - \frac{1}{\bar{z}}$, (g) $z^2 + z$, (h) z^3 .

[Check your results for $x = 2$, $y = 1$: (a) 4, (b) $i2$, (c) 5, (d) $\frac{3}{5} + i\frac{4}{5}$, (e) $\frac{4}{5}$, (f) $-i\frac{2}{5}$, (g) $5 + i5$, (h) $2 + i11$.]

Example Problem 5: Multiplication of complex numbers – geometrical interpretation [4]

Points: (a)[2](E); (b)[2](E)

- (a) Consider the polar representation, $z_j = (\rho_j \cos \phi_j, \rho_j \sin \phi_j)$, of two complex numbers, z_1 and z_2 , with $\phi_j \in [0, 2\pi)$. Show that multiplying them, $z_3 = z_1 z_2$, yields the relations $\rho_3 = \rho_1 \rho_2$ and $\phi_3 = (\phi_1 + \phi_2) \bmod(2\pi)$. [The $\bmod(2\pi)$ is needed since we restricted polar angles to lie in the interval $[0, 2\pi)$.] To this end, the following trigonometric identities are useful:

$$\begin{aligned}\cos \phi_1 \cos \phi_2 - \sin \phi_1 \sin \phi_2 &= \cos(\phi_1 + \phi_2), \\ \sin \phi_1 \cos \phi_2 + \cos \phi_1 \sin \phi_2 &= \sin(\phi_1 + \phi_2).\end{aligned}$$

- (b) For $z_1 = \sqrt{3} + i$, $z_2 = -2 + 2\sqrt{3}i$, compute the product $z_3 = z_1 z_2$, as well as $z_4 = 1/z_1$ and $z_5 = \bar{z}_1$. Find the polar representation (with $\phi \in [0, 2\pi)$) of all five complex numbers and sketch them in the complex plane (in one diagram). Is your result for z_3 consistent with (a)?

Example Problem 6: Differentiation of trigonometric functions [1]

Points: (a)[0,5](E); (b)[0,5](E).

Show that the trigonometric functions

$$\tan x = \frac{\sin x}{\cos x}, \quad \csc x = \frac{1}{\sin x}, \quad \sec x = \frac{1}{\cos x}, \quad \cot x = \frac{\cos x}{\sin x} = \frac{1}{\tan x},$$

satisfy the following identities:

$$(a) \quad \frac{d}{dx} \tan x = 1 + \tan^2 x = \sec^2 x, \quad (b) \quad \frac{d}{dx} \cot x = -1 - \cot^2 x = -\csc^2 x.$$

Example Problem 7: Differentiation of powers, exponentials, logarithms [2]

Points: [3](E).

Compute the first derivative of the following functions.

[Check your results against those in square brackets, where $[a, b]$ stands for $f'(a) = b$.]

$$\begin{array}{lll} (a) \quad f(x) = -\frac{1}{\sqrt{2x}} & [2, \frac{1}{8}] & (b) \quad f(x) = \frac{x^{1/2}}{(x+1)^{1/2}} \quad [3, \frac{1}{16\sqrt{3}}] \\ (c) \quad f(x) = e^x(2x-3) & [1, e] & (d) \quad f(x) = 3^x \quad [-1, \frac{\ln 3}{3}] \\ (e) \quad f(x) = x \ln x & [1, 1] & (f) \quad f(x) = x \ln(9x^2) \quad [\frac{1}{3}, 2] \end{array}$$

Example Problem 8: Differentiation of inverse trigonometric functions [4]

Points: (a)[1](E); (b)[1](M); (c)[2](M).

Compute the following derivatives of inverse trigonometric functions, f^{-1} . For each case, make a qualitative sketch showing $f(x)$ and $f^{-1}(x)$. If f is non-monotonic, consider domains with positive or negative slope separately. [Check your results: $[a, b]$ stands for $(f^{-1})'(a) = b$.]

$$(a) \quad \frac{d}{dx} \arcsin x \quad [\frac{1}{3}, \frac{3}{\sqrt{8}}] \quad (b) \quad \frac{d}{dx} \arccos x \quad [\frac{1}{2}, \frac{2}{\sqrt{3}}] \quad (c) \quad \frac{d}{dx} \arctan x \quad [1, \frac{1}{2}]$$

Hint: The identity $\sin^2 x + \cos^2 x = 1$ is useful for (a) and (b), $\sec^2 x = 1 + \tan^2 x$ for (c).

Example Problem 9: Integration by parts [6]

Points: [6](M)

Integrals of the form $I(z) = \int_{z_0}^z dx \, u(x)v'(x)$ can be written as $I(z) = [u(x)v(x)]_{z_0}^z - \int_{z_0}^z dx \, u'(x)v(x)$ using integration by parts. This is useful if $u'v$ can be integrated — either directly, or after further integrations by parts [see (b)], or after other manipulations [see (e,f)]. When doing such a calculation, it is advisable to clearly indicate the factors u , v' , v and u' . Always check that the derivative $I'(z) = dI/dz$ of the result reproduces the integrand! If a single integration by parts suffices to calculate $I(z)$, its derivative exhibits the cancellation pattern $I' = u'v + uv' - u'v = uv'$ [see (a,c,d)]; otherwise, more involved cancellations occur [see (b,e,f)].

Integrate the following integrals by parts. [Check your results against those in square brackets, where $[a, b]$ stands for $I(a) = b$.]

$$\begin{array}{lll} (a) \quad I(z) = \int_0^z dx \, x e^{2x} & [\frac{1}{2}, \frac{1}{4}] & (b) \quad I(z) = \int_0^z dx \, x^2 e^{2x} \quad [\frac{1}{2}, \frac{e}{8} - \frac{1}{4}] \\ (c) \quad I(z) = \int_0^z dx \, \ln x & [1, -1] & (d) \quad I(z) = \int_0^z dx \, \ln x \frac{1}{\sqrt{x}} \quad [1, -4] \\ (e) \quad I(z) = \int_0^z dx \, \sin^2 x & [\pi, \frac{\pi}{2}] & (f) \quad I(z) = \int_0^z dx \, \sin^4 x \quad [\pi, \frac{3\pi}{8}] \end{array}$$

Example Problem 10: Integration by substitution [4]

Points: [4](M)

Integrals of the form $I(z) = \int_{z_0}^z dx y'(x) f(y(x))$ can be written as $I(z) = \int_{y(z_0)}^{y(z)} dy f(y)$ by using the substitution $y = y(x)$, $dy = y'(x)dx$. When doing such integrals, it is advisable to explicitly write down $y(x)$ and dy , to ensure that you correctly identify the prefactor of $f(y)$. Always check that the derivative $I'(z) = dI/dz$ of the result reproduces the integrand! You'll notice that the factor $y'(z)$ emerges via the chain rule for differentiating composite functions.

Calculate the following integrals by substitution. [Check your results against those in square brackets, where $[a, b]$ stands for $I(a) = b$.]

$$\begin{array}{ll}
 \text{(a)} \quad I(z) = \int_0^z dx x \cos(x^2 + \pi) & \left[\sqrt{\frac{\pi}{2}}, -\frac{1}{2}\right] \\
 \text{(b)} \quad I(z) = \int_0^z dx \sin^3 x \cos x & \left[\frac{\pi}{4}, \frac{1}{16}\right] \\
 \text{(c)} \quad I(z) = \int_0^z dx \sin^3 x & \left[\frac{\pi}{3}, \frac{5}{24}\right] \\
 \text{(d)} \quad I(z) = \int_0^z dx \cosh^3 x & \left[\ln 2, \frac{57}{64}\right] \\
 \text{(e)} \quad I(z) = \int_0^z dx \frac{\sqrt{1 + \ln(x+1)}}{x+1} & \left[e^3 - 1, \frac{14}{3}\right] \\
 \text{(f)} \quad I(z) = \int_0^z dx x^3 e^{-x^4} & \left[\sqrt[4]{\ln 2}, \frac{1}{8}\right]
 \end{array}$$

[Total Points for Example Problems: 34]

Homework Problem 1: Composition of maps [2]

Points: (a)[0,5](E); (b)[0,5](E); (c)[0,5](E); (d)[0,5](E).

- (a) Consider the set $S = \{-2, -1, 0, 1, 2\}$. Find its image, $T = A(S)$, under the map $n \mapsto A(n) = n^2$. Is the map $A : S \rightarrow T$ surjective? Injective? Bijective?
- (b) Find the image, $U = B(T)$, of the set T from part (a) under the map $n \mapsto B(n) = \sqrt{n}$.
- (c) Find the composite map $C = B \circ A$.
- (d) Which of the above maps A , B and C are surjective? Injective? Bijective?

Homework Problem 2: The groups of addition modulo 5 and rotations by multiples of 72 deg [3]

Points: (a)[1](E); (b)[1](E); (c)[0,5](E); (d)[0,5](E).

- (a) Consider the set $\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$, endowed with the group operation

$$+ : \mathbb{Z}_5 \times \mathbb{Z}_5 \rightarrow \mathbb{Z}_5, \quad (p, p') \mapsto p + p' \equiv (p + p') \bmod 5.$$

Set up the composition table for the group $(\mathbb{Z}_5, +)$. Which element is the neutral element? For a given $n \in \mathbb{Z}$, which element is the inverse of n ?

- (b) Let $r(\phi)$ denote a rotation by ϕ degrees about a fixed axis, with $r(\phi + 360) = r(\phi)$. Consider the set of rotations by multiples of 72 deg,

$$\mathcal{R}_{72} = \{r(0), r(72), r(144), r(216), r(288)\},$$

and the group $(\mathcal{R}_{72}, \cdot)$, where the group operation $\cdot$ involves two rotations in succession:

$$\cdot : \mathcal{R}_{72} \times \mathcal{R}_{72} \rightarrow \mathcal{R}_{72}, \quad (r(\phi), r(\phi')) \mapsto r(\phi) \cdot r(\phi') \equiv r(\phi + \phi').$$

Set up the multiplication table for this group. Which element is the neutral element? Which element is the inverse of $r(\phi)$?

- (c) Explain why the groups $(\mathbb{Z}_5, +)$ and $(\mathcal{R}_{72}, \cdot)$ are isomorphic.
- (d) Let $(\mathbb{Z}_n, +)$ denote the group of integer addition modulo n of the elements of the set $\mathbb{Z}_n = \{0, 1, \dots, n-1\}$. Which group of discrete rotations is isomorphic to this group?

Homework Problem 3: Decomposing permutations into sequences of pair permutations [2]

Consider the permutation group S_n . Any permutation can be decomposed into a sequence of **pair permutations**, i.e. permutations which exchange just two objects, leaving the others unchanged. Examples:

$$\begin{aligned} 123 &\xrightarrow{[321]} 321 \xrightarrow{[132]} 231 && \Rightarrow [231] = [132] \circ [321]. \\ 1234 &\xrightarrow{[2134]} 2134 \xrightarrow{[3214]} 2314 && \Rightarrow [2314] = [3214] \circ [2134], \\ 1234 &\xrightarrow{[3214]} 3214 \xrightarrow{[1324]} 2314 && \Rightarrow [2314] = [1324] \circ [3214], \\ 1234 &\xrightarrow{[4231]} 4231 \xrightarrow{[1432]} 2431 \xrightarrow{[1243]} 2341 \xrightarrow{[4231]} 2314 && \Rightarrow [2314] = [4231] \circ [1243] \circ [1432] \circ [4231]. \end{aligned}$$

The last three lines illustrate that a given permutation can be pair-decomposed in several ways, and that these may or may not involve different numbers of pair exchanges. However, one may convince oneself (try it!) that all pair decompositions of a given permutation have the same **parity**, i.e. the number of exchanges is either always **even** or always **odd**.

To find a 'minimal' (shortest possible) pair decomposition of a given permutation, say $[2413]$, we may start from the naturally-ordered string 1234 and rearrange it to its desired form, 2413, one pair permutation at a time, bringing the 2 to the first slot, then the 4 to the second slot, etc. This yields $1234 \xrightarrow{[2134]} 2134 \xrightarrow{[4231]} 2431 \xrightarrow{[3214]} 2413$, hence $[2413] = [3214] \circ [4231] \circ [2134]$.

Find a minimal pair decomposition and the parity of each of the following permutations:

- (a) $[132]$, (b) $[231]$, (c) $[3412]$, (d) $[3421]$, (e) $[15234]$, (f) $[31542]$.

Homework Problem 4: Algebraic manipulations with complex numbers [3]

Points: (a)[1](E); (b)[1](M); (c)[1](E).

For $z = x + iy \in \mathbb{C}$, bring each of the following expressions into standard form:

$$(a) (z + i)^2, \quad (b) \frac{z}{z + 1}, \quad (c) \frac{\bar{z}}{z - i}.$$

[Check your results for $x = 1, y = 2$: (a) $-8 + i6$, (b) $\frac{3}{4} + i\frac{1}{4}$, (c) $-\frac{1}{2} - i\frac{3}{2}$.]

Homework Problem 5: Multiplication of complex numbers – geometrical interpretation [2]

Points: [2](E)

For $z_1 = \frac{1}{\sqrt{8}} + \frac{1}{\sqrt{8}}i$, $z_2 = \sqrt{3} - i$, compute the product $z_3 = z_1 z_2$, as well as $z_4 = 1/z_1$ and $z_5 = \bar{z}_1$. Find the polar representation (with $\phi \in [0, 2\pi)$) of all five complex numbers and sketch them in the complex plane (in one diagram).

Homework Problem 6: Differentiation of hyperbolic functions [2]

Points: (a)[0,5](E); (b,c)[0,5](E); (d)[0,5](E); (e)[0,5](E).

Show that the hyperbolic functions

$$\begin{aligned} \sinh x &= \frac{1}{2}(e^x - e^{-x}), & \cosh x &= \frac{1}{2}(e^x + e^{-x}), & \tanh x &= \frac{\sinh x}{\cosh x}, \\ \operatorname{csch} x &= \frac{1}{\sinh x}, & \operatorname{sech} x &= \frac{1}{\cosh x}, & \coth x &= \frac{\cosh x}{\sinh x} = \frac{1}{\tanh x}, \end{aligned}$$

satisfy the following identities:

- (a) $\cosh^2 x - \sinh^2 x = 1$,
 (b) $\frac{d}{dx} \sinh x = \cosh x$, (c) $\frac{d}{dx} \cosh x = \sinh x$.
 (d) $\frac{d}{dx} \tanh x = 1 - \tanh^2 x = \operatorname{sech}^2 x$, (e) $\frac{d}{dx} \coth x = 1 - \coth^2 x = -\operatorname{csch}^2 x$.

Homework Problem 7: Differentiation of powers, exponentials, logarithms [2]

Points: [2](E) (Solve any 4 subproblems; beyond that: 0.25 bonus per subproblem.)

Compute the first derivative of the following functions.

[Check your results against those in square brackets, where $[a, b]$ stands for $f'(a) = b$.]

- (a) $f(x) = \sqrt[3]{x^2}$ $[8, \frac{1}{3}]$ (b) $f(x) = \frac{x}{(x^2 + 1)^{1/2}}$ $[1, \frac{1}{\sqrt{8}}]$
 (c) $f(x) = -e^{(1-x^2)}$ $[1, 2]$ (d) $f(x) = 2^{x^2}$ $[1, 4 \ln 2]$
 (e) $f(x) = 2 \frac{\sqrt{\ln x}}{x}$ $[e, -\frac{1}{e^2}]$ (f) $f(x) = \ln \sqrt{x^2 + 1}$ $[1, \frac{1}{2}]$

Homework Problem 8: Differentiation of inverse hyperbolic functions [2]

Points: [2](M) (Solve subproblem, (b); beyond that: 0.5 bonus points per subproblem.)

Compute the following derivatives of inverse hyperbolic functions, f^{-1} . For each case, make a qualitative sketch showing $f(x)$ and $f^{-1}(x)$. If f is non-monotonic, consider domains with positive or negative slope separately. [Check your results: $[a, b]$ stands for $(f^{-1})'(a) = b$.]

- (a) $\frac{d}{dx} \operatorname{arcsinh} x$ $[2, \frac{1}{\sqrt{5}}]$ (b) $\frac{d}{dx} \operatorname{arccosh} x$ $[2, \frac{1}{\sqrt{3}}]$ (c) $\frac{d}{dx} \operatorname{arctanh} x$ $[\frac{1}{2}, \frac{4}{3}]$

Hint: The identity $\cosh^2 x = 1 + \sinh^2 x$ is useful for (a) and (b), $\operatorname{sech}^2 x = 1 - \tanh^2 x$ for (c).

Homework Problem 9: Integration by parts [4]

Points: [4](M) (Solve any 4 subproblems; beyond that: 0.5 bonus per subproblem.)

Integrate the following integrals by parts. [Check your results against those in square brackets, where $[a, b]$ stands for $I(a) = b$.]

- (a) $I(z) = \int_0^z dx \, x \sin(2x)$ $[\frac{\pi}{2}, \frac{\pi}{4}]$ (b) $I(z) = \int_0^z dx \, x^2 \cos(2x)$ $[\frac{\pi}{2}, -\frac{\pi}{4}]$

$$\begin{array}{ll}
\text{(c) } I(z) = \int_0^z dx (\ln x) x & \left[1, -\frac{1}{4}\right] \\
\text{(e) } I(z) = \int_0^z dx \cos^2 x & \left[\pi, \frac{\pi}{2}\right]
\end{array}
\qquad
\begin{array}{ll}
\text{(d) } I(z) \stackrel{[n > -1]}{=} \int_0^z dx (\ln x) x^n & \left[1, \frac{-1}{(n+1)^2}\right] \\
\text{(f) } I(z) = \int_0^z dx \cos^4 x & \left[\pi, \frac{3}{8}\pi\right]
\end{array}$$

Homework Problem 10: Integration by substitution [3]

Points: [3](M) (Solve any 3 subproblems; beyond that: 0.5 bonus per subproblem.)

Calculate the following integrals by substitution. [Check your results versus those in square brackets, where $[a, b]$ stands for $I(a) = b$.]

$$\begin{array}{ll}
\text{(a) } I(z) = \int_0^z dx x^2 \sqrt{x^3 + 1} & \left[2, \frac{52}{9}\right] \\
\text{(c) } I(z) = \int_0^z dx \cos^3 x & \left[\frac{\pi}{4}, \frac{5}{6\sqrt{2}}\right] \\
\text{(e) } I(z) = \int_0^z dx \frac{\sin \sqrt{\pi x}}{\sqrt{x}} & \left[\frac{\pi}{9}, \frac{1}{\sqrt{\pi}}\right]
\end{array}
\qquad
\begin{array}{ll}
\text{(b) } I(z) = \int_0^z dx \sin x e^{\cos x} & \left[\frac{\pi}{3}, e - \sqrt{e}\right] \\
\text{(d) } I(z) = \int_0^z dx \sinh^3 x & \left[\ln 3, \frac{44}{81}\right] \\
\text{(f) } I(z) = \int_0^z dx \sqrt{x} e^{\sqrt{x^3}} & \left[(\ln 4)^{2/3}, 2\right]
\end{array}$$

[Total Points for Homework Problems: 25]
