# Sheet 01: Mathematical Foundations

## Solution Optional Problem 1: Group of discrete translations in one dimension [4]

(a) Consider the group axioms:

- (i) Closure: The integers are closed under usual addition: $m, n \in \mathbb{Z} \Rightarrow m + n \in \mathbb{Z}$. All $x, y \in \mathbb{G}$ are integer multiples of $\lambda$, hence there exist integers $n_x, n_y \in \mathbb{Z}$ such that $x = \lambda n_x, y = \lambda n_y$. It follows that $T(x, y) = x + y = \lambda \cdot n_x + \lambda \cdot n_y = \lambda \cdot (n_x + n_y) \in \lambda \cdot \mathbb{Z} = \mathbb{G}$. ✓
- (ii) Associativity: The usual addition rule for real numbers is associative: $a, b, c \in \mathbb{R} \Rightarrow (a+b)+c = a+(b+c)$. For $x, y, z \in \mathbb{G}$ we therefore have $T(T(x, y), z) = T(x+y, z) = (x + y) + z = x + (y + z) = T(x, y + z) = T(x, T(y, z))$. ✓
- (iii) Neutral element: The neutral element is $0 = \lambda \cdot 0 \in \mathbb{G}$: For all $x \in \mathbb{G}$ we have: $T(x, 0) = x + 0 = x$. ✓
- (iv) Inverse element: The inverse element of $n \in \mathbb{Z}$ is $-n \in \mathbb{Z}$. Thus the inverse of $x = \lambda \cdot n \in \mathbb{G}$ is $-x \equiv \lambda \cdot (-n) \in \mathbb{G}$, since $T(x, -x) = \lambda \cdot n + \lambda \cdot (-n) = \lambda \cdot (n + (-n)) = \lambda \cdot 0 = 0$. ✓
- (v) Commutativity (for the group to be abelian): For all $x, y \in \mathbb{G}$ we have $T(x, y) = x + y = y + x = T(y, x)$, since the usual addition of real numbers is commutative. ✓

Since $(\mathbb{G}, T)$ satisfies properties (i)-(v), it is an abelian group. ✓
*Remark:* For $\lambda = 1$, the group $(\mathbb{G}, T)$ is identical to $(\mathbb{Z}, +)$.

(b) The group axioms of $(\mathbb{T}, \boldsymbol{+})$ follow directly from those of $(\mathbb{G}, T)$:

- (i) Closure: $\mathcal{T}_x, \mathcal{T}_y \in \mathbb{T} \Rightarrow \mathcal{T}_x \boldsymbol{+} \mathcal{T}_y = \mathcal{T}_{T(x,y)} \in \mathbb{T}$, since if $x, y \in \mathbb{G}$, then $T(x, y) \in \mathbb{G}$ [see (a)]. ✓
- (ii) Associativity: For $\mathcal{T}_x, \mathcal{T}_y, \mathcal{T}_z \in \mathbb{T}$ we have: $(\mathcal{T}_x \boldsymbol{+} \mathcal{T}_y) \boldsymbol{+} \mathcal{T}_z = \mathcal{T}_{T(x,y)} \boldsymbol{+} \mathcal{T}_z = \mathcal{T}_{T(T(x,y),z)} \overset{\text{(a)}}{=} \mathcal{T}_{T(x,T(y,z))} = \mathcal{T}_x \boldsymbol{+} \mathcal{T}_{T(y,z)} = \mathcal{T}_x \boldsymbol{+} (\mathcal{T}_y \boldsymbol{+} \mathcal{T}_z)$. ✓
- (iii) Neutral element: The neutral element is $\mathcal{T}_0 \in \mathbb{T}$: For all $\mathcal{T}_x \in \mathbb{T}$ we have: $\mathcal{T}_x \boldsymbol{+} \mathcal{T}_0 = \mathcal{T}_{T(x,0)} = \mathcal{T}_{x+0} = \mathcal{T}_x$. ✓
- (iv) Inverse element: The inverse element of $\mathcal{T}_x \in \mathbb{T}$ is $\mathcal{T}_{-x} \in \mathbb{T}$, where $-x$ is the inverse element of $x \in \mathbb{G}$ with respect to $T$, since $\mathcal{T}_x \boldsymbol{+} \mathcal{T}_{-x} = \mathcal{T}_{T(x,-x)} = \mathcal{T}_{x+(-x)} = \mathcal{T}_0$. ✓
- (v) Commutativity (for the group to be abelian): For all $x, y \in \mathbb{G}$ we have $\mathcal{T}_x \boldsymbol{+} \mathcal{T}_y = \mathcal{T}_{T(x,y)} = \mathcal{T}_{T(y,x)} = \mathcal{T}_y \boldsymbol{+} \mathcal{T}_x$, since the composition rule $T$ in $\mathbb{G}$ is commutative. ✓

Since $(\mathbb{T}, \boldsymbol{+})$ satisfies properties (i)-(v), it is an abelian group. ✓

**Solution Optional Problem 2: Group of discrete translations on a ring [4]**

(a) Consider the group axioms:

- (i) Closure: by definition $a, b \in \mathbb{Z} \Rightarrow (a+b) \bmod N \in \mathbb{Z} \bmod N$. Thus: $x, y \in \mathbb{G} \Rightarrow \exists n_x, n_y \in \mathbb{Z} \bmod N : x = \lambda n_x, y = \lambda n_y$. It follows that $T(x,y) = \lambda \cdot (n_x + n_y) \bmod N \in \lambda \cdot \mathbb{Z} \bmod N = \mathbb{G}$. ✓
- (ii) Associativity: The usual addition of integers is associative, $m, n, l \in \mathbb{Z} \Rightarrow (m+n)+l = m+(n+l)$, and this property remains true for addition modulo $N$. For $x, y, z \in \mathbb{G}$ we therefore have: $T(T(x,y),z) = \lambda \cdot ((n_x + n_y) + n_z)(\bmod N) = \lambda \cdot (n_x + (n_y + n_z))(\bmod N) = T(x, T(y,z))$. ✓
- (iii) Neutral element: The neutral element is $0 = \lambda \cdot 0 \in \mathbb{G}$: For all $x \in \mathbb{G}$ we have: $T(x, 0) = \lambda \cdot (n_x + 0)(\bmod N) = x$. ✓
- (iv) Inverse element: The inverse element of $n \in \mathbb{Z} \bmod N$ is $[N + (-n)] \bmod N \in \mathbb{Z} \bmod N$. Therefore the inverse element of $x = \lambda \cdot n \in \mathbb{G}$ is given by $-x \equiv \lambda \cdot (N + (-n)) \in \mathbb{G}$, since $T(x, -x) = \lambda \cdot (n + (N + (-n)))(\bmod N) = \lambda \cdot 0 (\bmod N) = 0$. ✓
- (v) Commutativity (for the group to be abelian): For all $x, y \in \mathbb{G}$ we have $T(x,y) = \lambda \cdot (n_x + n_y) \bmod N = \lambda \cdot (n_y + n_x) \bmod N = T(y,x)$, since the usual addition of real numbers is commutative, and this property remains true for addition modulo $N$. ✓

Since $(\mathbb{G}, T)$ satisfies properties (i)-(v), it is an abelian group. ✓

(b) The group axioms of $(\mathbb{T}, \boldsymbol{+})$ follow directly from those of $(\mathbb{G}, T)$:

- (i) Closure: $\mathcal{T}_x, \mathcal{T}_y \in \mathbb{T} \Rightarrow \mathcal{T}_x \boldsymbol{+} \mathcal{T}_y = \mathcal{T}_{T(x,y)} \in \mathbb{T}$, since $x, y \in \mathbb{G} \Rightarrow T(x,y) \in \mathbb{G}$ [see (a)]. ✓
- (ii) Associativity: For $\mathcal{T}_x, \mathcal{T}_y, \mathcal{T}_z \in \mathbb{T}$ we have: $(\mathcal{T}_x \boldsymbol{+} \mathcal{T}_y) \boldsymbol{+} \mathcal{T}_z = \mathcal{T}_{T(x,y)} \boldsymbol{+} \mathcal{T}_z = \mathcal{T}_{T(T(x,y),z)} \overset{\text{(a)}}{=} \mathcal{T}_{T(x,T(y,z))} = \mathcal{T}_x \boldsymbol{+} \mathcal{T}_{T(y,z)} = \mathcal{T}_x \boldsymbol{+} (\mathcal{T}_y \boldsymbol{+} \mathcal{T}_z)$. ✓
- (iii) Neutral element: The neutral element is $\mathcal{T}_0 \in \mathbb{T}$: For all $\mathcal{T}_x \in \mathbb{T}$ we have: $\mathcal{T}_x \boldsymbol{+} \mathcal{T}_0 = \mathcal{T}_{T(x,0)} = \mathcal{T}_{x+0} = \mathcal{T}_x$. ✓
- (iv) Inverse element: The inverse element of $\mathcal{T}_x \in \mathbb{T}$ is $\mathcal{T}_{-x} \in \mathbb{T}$, where $-x$ is the inverse element of $x \in \mathbb{G}$ with respect to $T$, since $\mathcal{T}_x \boldsymbol{+} \mathcal{T}_{-x} = \mathcal{T}_{T(x,-x)} = \mathcal{T}_{x+(-x)} = \mathcal{T}_0$. ✓
- (v) Commutativity (for the group to be abelian): For all $x, y \in \mathbb{G}$ we have $\mathcal{T}_x \boldsymbol{+} \mathcal{T}_y = \mathcal{T}_{T(x,y)} = \mathcal{T}_{T(y,x)} = \mathcal{T}_y \boldsymbol{+} \mathcal{T}_x$, since the composition rule $T$ in $\mathbb{G}$ is commutative. ✓

Since $(\mathbb{T}, \boldsymbol{+})$ satisfies properties (i)-(v), it is an abelian group. ✓

**Solution Optional Problem 3: L'Hôpital's rule [4]**

For (a,b) we may apply L'Hôpital's rule in the form $\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}$, since the given functions $f$ and $g$ both vanish at the limiting point $x_0$, whereas $f'$ and $g'$ are finite there:

(a) $$\lim_{x \to 1} \frac{x^2 + (a-1)x - a}{x^2 + 2x - 3} = \lim_{x \to 1} \frac{2x + (a-1)}{2x + 2} = \frac{2 + (a-1)}{2+2} = \boxed{\frac{a+1}{4}}.$$

(b) $\lim_{x\to 0} \frac{\sin(ax)}{x+ax^2} = \lim_{x\to 0} \frac{a\cos(ax)}{1+2ax} = \boxed{a}$.

(c) We use $\lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{f'(x)}{g'(x)} = \lim_{x\to 0}\frac{f''(x)}{g''(x)}$, since not only $f$ and $g$, but also $f'$ and $g'$ vanish at $x = 0$.

$$\lim_{x\to 0} \frac{1-\cos(ax)}{\sin^2 x} = \lim_{x\to 0} \frac{a\sin(ax)}{2\sin x\cos x} = \lim_{x\to 0} \frac{a^2\cos(ax)}{2[\cos^2 x - \sin^2 x]} = \boxed{\frac{a^2}{2}}.$$

(d) We use L'Hôpital's rule three times, since $f^{(n)}(0)$ and $g^{(n)}(0)$ all vanish for $n = 0, 1, 2$:

$$\lim_{x\to 0} \frac{x^3}{\sin(ax)-ax} = \lim_{x\to 0} \frac{3x^2}{a\cos(ax)-a} = \lim_{x\to 0} \frac{6x}{-a^2\sin(ax)} = \lim_{x\to 0} \frac{6}{-a^3\cos(ax)} = \boxed{\frac{-6}{a^3}}.$$

(e) The naive answer, $\lim_{x\to 0}(x\ln x) \stackrel{?}{=} 0\cdot\infty$, is ill-defined, hence we evoke L'Hôpital's rule for the case $\lim_{x\to 0}|f(x)| = \lim_{x\to 0}|g(x)| = \infty$, with $f(x) = \ln x$ and $g(x) = x^{-1}$:

$$\lim_{x\to 0}(x\ln x) = \lim_{x\to 0}\frac{\ln x}{x^{-1}} = \lim_{x\to 0}\frac{x^{-1}}{-x^{-2}} = \lim_{x\to 0} -x = \boxed{0}.$$

**Solution Optional Problem 4: L'Hôpital's rule [4]**

We use L'Hôpital's rule, $\lim\limits_{x\to x_0}\frac{f(x)}{g(x)} = \lim\limits_{x\to x_0}\frac{f'(x)}{g'(x)}$, once for (a,b), twice for (c), four times for (d):

(a) $\lim_{x\to a} \frac{x^2+(2-a)x-2a}{x^2-(a+1)x+a} = \lim_{x\to a}\frac{2x+(2-a)}{2x-(a+1)} = \frac{2a+(2-a)}{2a-(a+1)} = \boxed{\frac{a+2}{a-1}}$.

(b) $\lim_{x\to 0}\frac{\sinh(x)}{\tanh(ax)} = \lim_{x\to 0}\frac{\cosh(x)}{a\,\mathrm{sech}^2(ax)} = \boxed{\frac{1}{a}}$.

(c) $\lim_{x\to 0}\frac{e^{x^2}-1}{(e^{ax}-1)^2} = \lim_{x\to 0}\frac{2xe^{x^2}}{2a(e^{ax}-1)e^{ax}} = \lim_{x\to 0}\frac{2(1+2x)e^{x^2}}{2a^2(2e^{2ax}-e^{ax})} = \boxed{\frac{1}{a^2}}$.

(d) $\lim_{x\to 0}\frac{\cosh(ax)+\cos(ax)-2}{x^4} = \lim_{x\to 0} a\frac{\sinh(ax)-\sin(ax)}{4x^3} = \lim_{x\to 0} a^2\frac{\cosh(ax)-\cos(ax)}{4\cdot 3x^2}$

$$= \lim_{x\to 0} a^3\frac{\sinh(ax)+\sin(ax)}{4\cdot 3\cdot 2x} = \lim_{x\to 0} a^4\frac{\cosh(ax)+\cos(ax)}{4\cdot 3\cdot 2} = a^4\frac{1+1}{24} = \boxed{\frac{a^4}{12}}.$$

(e) For $\alpha \le 0$ the statement is trivially true, since then both $\ln^\alpha(x)$ and $x^\beta$ vanish for $x\to 0$. We thus focus on the case $\alpha > 0$. Then the naive answer, $\lim_{x\to 0}(x^\beta \ln^\alpha x) \stackrel{?}{=} 0\cdot\infty$, is ill-defined, hence we evoke L'Hôpital's rule for the case $\lim_{x\to 0}|f(x)| = \lim_{x\to 0}|g(x)| = \infty$, with $f(x) = \ln^\alpha x$ and $g(x) = x^{-\beta}$:

$$\lim_{x\to 0}(x^\beta\ln^\alpha x) = \lim_{x\to 0}\frac{\ln^\alpha x}{x^{-\beta}} = \lim_{x\to 0}\frac{\alpha(\ln x)^{\alpha-1}x^{-1}}{-\beta x^{-\beta-1}} = -\frac{\alpha}{\beta}\lim_{x\to 0}(x^\beta \ln^{\alpha-1} x).$$

The final expression has a similar form as the initial one, but the power of the logarithm has been reduced by one. Repeating this procedure, we find $\lim_{x\to 0} \propto (x^\beta \ln^{\alpha-n} x)$ after $n$ steps, which evidently equals $\boxed{0}$ once $n$ has become larger than $\alpha$.

[Total Points for Optional Problems: 16]