Problem 1

(1.a) Introduce Pauli operators $\sigma_x = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$, $\sigma_y = \begin{pmatrix} & -i \\ i & \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$

$b_\alpha = \frac{1}{2}(\sigma^x_\alpha + i\sigma^y_\alpha)$, $b^\dagger_\alpha = \frac{1}{2}(\sigma^x_\alpha - i\sigma^y_\alpha)$

Pauli operators have commutation and anti-commutation relations

$[\sigma^j, \sigma^k] = 2i\sum_l \epsilon_{jkl}\sigma_l$, $\{\sigma^j, \sigma^k\} = 2\delta_{jk} 1$

Remind that tensor product have $(A\otimes B)\cdot(C\otimes D) = AC\otimes BD$

If $[B,D]=0$, commutator and anti-commutator factorize as

$[A\otimes B, C\otimes D] = [A,C]\otimes BD$, $\{A\otimes B, C\otimes D\} = \{A,C\}\otimes BD$

Consider Pauli string, $\sigma^k_\alpha = 1\otimes 1\otimes\cdots\otimes \underset{\alpha}{\sigma^k}\otimes\cdots\otimes 1\otimes\cdots$

We have $[\sigma^k_\alpha, \sigma^j_\beta] = \delta_{\alpha\beta}\cdot[\sigma^k_\alpha, \sigma^j_\alpha]$

$[b_\alpha, b_\beta] = \delta_{\alpha\beta}[b_\alpha, b_\alpha] = 0$, $[b_\alpha, b^\dagger_\beta] = \delta_{\alpha\beta}[b_\alpha, b^\dagger_\alpha] = \delta_{\alpha\beta}\cdot\frac{1}{4}(i[\sigma^y_\alpha, \sigma^x_\alpha] - i[\sigma^x_\alpha, \sigma^y_\alpha])$.

$= \delta_{\alpha\beta}\sigma^z_\alpha = \delta_{\alpha\beta} e^{i\pi n_\alpha}$

(1.a) $c_\alpha = U_\alpha b_\alpha = \prod^{\otimes}_{a<\alpha}\sigma^z_a \otimes b_\alpha\otimes\cdots = \sigma^z\otimes\sigma^z\otimes\cdots\otimes b\otimes 1\otimes\cdots\otimes 1$.

Since $[\sigma^z, \sigma^z] = 0$, $[1, \hat{O}] = 0$, assume $\alpha<\beta$

$c_\alpha = \sigma_z\otimes\sigma_z\otimes\cdots\otimes b\otimes 1\otimes\cdots\otimes 1\otimes\cdots$

$c_\beta = \sigma_z\otimes\sigma_z\otimes\cdots\otimes\sigma_z\otimes\sigma_z\otimes\cdots\otimes b\otimes\cdots$

$\{c_\alpha, c_\beta\} = 1\otimes\cdots\otimes 1\otimes\{b, \sigma_z\}\otimes 1\cdots\otimes b\otimes\cdots$

$\{b, \sigma^z\} = \frac{1}{2}(\{\sigma_x, \sigma^z\} + i\{\sigma_y, \sigma^z\}) = 0$

When $\alpha>\beta$. $\{c_\alpha, c_\beta\} = \{c_\beta, c_\alpha\}$.

When $\alpha=\beta$, $\{b, b\} = \frac{1}{4}(\{\sigma^x, \sigma^x\} - \{\sigma^y, \sigma^y\}) = 0 \rightarrow \{c_\alpha, c_\beta\} = 0$

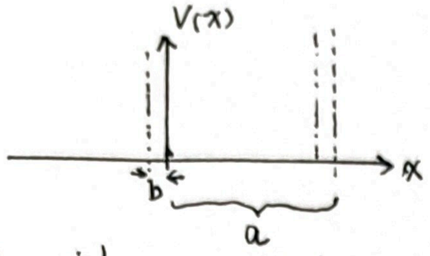
For $\{c^\dagger_\alpha, c^\dagger_\beta\}$, we similarly have. $\{b^\dagger, \sigma_z\} = 0$ and $\{b^\dagger, b^\dagger\} = 0 \rightarrow \{c^\dagger_\alpha, c^\dagger_\beta\} = 0$

For $\{c_\alpha, c^\dagger_\beta\}$, we have $\{b, b^\dagger\} = \frac{1}{4}(\{\sigma^x, \sigma^x\} + \{\sigma^y, \sigma^y\}) = 1 \rightarrow \{c_\alpha, c^\dagger_\beta\} = \delta_{\alpha\beta}$

Phase string $\sigma^z\otimes\cdots\otimes\sigma^z$ imply a non-local interaction.

Problem 2

(2.a)



Region with $V=0$, $(0, a-b)$

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi = E\psi \mapsto \psi = e^{-i\alpha x} \quad \text{where } \alpha^2 = \frac{2mE}{\hbar^2}, \quad \psi^A = A_+ e^{+i\alpha x} + A_- e^{-i\alpha x}$$

Region with $V_b = V_0/b$ $(a-b, a)$

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi + V_b\psi = E\psi \mapsto \psi = e^{-i\beta x} \quad \text{where } \beta^2 = \frac{2m(E-V_b)}{\hbar^2}, \quad \psi^B = B_+ e^{+i\beta x} + B_- e^{-i\beta x}$$

With Bloch theorem

$$\psi(x) = e^{ikx} u(x)$$

And boundary condition (periodic)

$$u^A(0) = u^B(0), \quad u'_A(0) = u'_B(0)$$

$$u_A(a-b) = u_B(-b), \quad u'_A(a-b) = u'_B(-b)$$

These conditions yield the following linear system

$$\begin{pmatrix} 1 & 1 & -1 & -1 \\ \alpha & -\alpha & -\beta & \beta \\ e^{i(\alpha-k)(a-b)} & e^{-i(\alpha+k)(a-b)} & -e^{-i(\beta+k)b} & -e^{i(\beta+k)b} \\ (\alpha-k)e^{i(\alpha-k)(a-b)} & -(\alpha+k)e^{-i(\alpha+k)(a-b)} & -(\beta-k)e^{-i(\beta-k)b} & (\beta+k)e^{i(\beta+k)b} \end{pmatrix} \begin{pmatrix} A_+ \\ A_- \\ B_+ \\ B_- \end{pmatrix} = \vec{0} \qquad Ax = 0$$

To get a non-trivial solution, the determinant has to be zero

$$|A| = 4(\alpha^2+\beta^2)\, e^{-ik(a-b)} e^{ikb} \sin(\alpha(a-b))\sin(\beta b) + 4\alpha\beta\left[e^{2ikb}(1+e^{-2ika}) - 2e^{-ik(a-b)} e^{ikb}\cos(\alpha(a-b))\cos(\beta b)\right] = 0$$

$$\underbrace{\frac{1}{2}\left(e^{ika} + e^{-ika}\right)}_{\cos(ka)} = \cos(\alpha(a-b))\cos(\beta b) - \frac{\alpha^2+\beta^2}{2\alpha\beta}\sin(\alpha(a-b))\sin(\beta b).$$

Take the limit $b \to 0$, $\beta b \to 0$, $\beta^2 b = -\frac{2mV_0}{\hbar^2}$, $\alpha^2 b = \frac{2mEb}{\hbar^2} = 0$

$$\cos(ka) = \cos(\alpha a) + \frac{v}{2\alpha}\sin(\alpha a)$$

where $v \equiv \frac{2mV_0}{\hbar^2}$ and $\alpha^2 = \frac{2mE}{\hbar^2}$

(2.b)

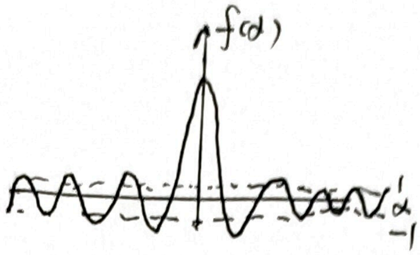
$\cos(ka) \in [-1, 1]$

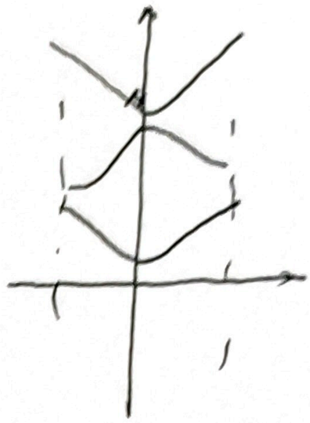
$f(\alpha) = \cos(\alpha a) + \frac{v}{2\alpha}\sin(\alpha a)$

have values out of $[-1, 1]$

there fore for some $E \to \alpha$,

no k_s are corresponding to them





(2b)

$V_0 \to 0$, free particle.

$V_0 \to \infty$, particle in a box

(2.c) $\rho(E) = \frac{1}{2\pi}\int \frac{dS_E}{|\nabla_k E|} = \frac{1}{\pi}\left|\frac{dk}{dE}\right|$

$\frac{dk}{dE} = \frac{dk}{d\alpha}\cdot\frac{d\alpha}{dE}$ $\qquad \frac{d\alpha}{dE} = \frac{1}{\hbar}\sqrt{\frac{m}{2E}}$ $\qquad \frac{d}{dx}\arccos x = \frac{-1}{\sqrt{1-x^2}}$

$$\frac{dk}{d\alpha} = \frac{\sin(\alpha a) + va(\sin(\alpha a)/(\alpha a)^2 - \cos(\alpha a)/(\alpha a))}{\sqrt{1 - (va\frac{\sin(\alpha a)}{\alpha a} + \cos(\alpha a))^2}}$$