Ex.3

Problem 1

$H_{tot} = H_n + H + H_{int}$, $H_{int} = \alpha \sum_j \delta(\vec{R} - \vec{r_j}) \vec{S_n} \cdot \vec{S_j}$

neutron scattering $|\vec{p}\rangle|\uparrow\rangle_n \rightarrow |\vec{p}'\rangle|\downarrow\rangle_n$

$|\Psi_i\rangle = |\vec{p}\uparrow\rangle_n \otimes |\phi_i\rangle$

$|\Psi_f\rangle = |\vec{p}'\downarrow\rangle_n \otimes |\phi_f\rangle$ $\qquad |\vec{p}\uparrow\rangle = \frac{1}{\sqrt{L^3}} e^{i\vec{p}\cdot\vec{R}} \otimes |\uparrow\rangle_n$

$\varepsilon_i = \frac{\hbar^2 p^2}{2M_n} + E_i$, $\varepsilon_f = \frac{\hbar^2 p'^2}{2M_n} + E_f$

$\Gamma(\vec{p}\uparrow, \vec{p}'\downarrow) = \sum_i \frac{e^{-\beta E_i}}{Z} \cdot \Gamma_{if}$, $\Gamma_{if} = \frac{2\pi}{\hbar} \delta(E_f - E_i - \hbar\omega) |M_{if}|^2$

where, $\hbar\omega = \frac{1}{2M_n}(p^2 - p'^2)$, $M_{if} = \langle \Psi_f | \hat{H}_{int} | \Psi_i \rangle$

$= \langle \vec{p}'\downarrow \phi_f | \alpha \sum_j \delta(\vec{R} - \vec{r_j}) \vec{S_n} \cdot \vec{S_j} | \vec{p}\uparrow \phi_i \rangle$

$\langle \vec{p}' | \alpha \sum_j \delta(\vec{R} - \vec{r_j}) | \vec{p} \rangle = \frac{\alpha}{L^3} \sum_j e^{-i\vec{q}\cdot\vec{r_j}}$, $\quad \langle \downarrow_n | \vec{S_n} \cdot \vec{S_j} | \uparrow_n \rangle = S_j^+$

$\vec{S_n} \cdot \vec{S_j} = S_n^+ S_j^- + S_n^- S_j^+ + S_n^z S_j^z$

$M_{if} = \frac{\alpha}{L^3} \sum_j \langle \phi_f | e^{-i\vec{q}\cdot\vec{r_j}} \cdot S_j^+ | \phi_i \rangle$

$\Gamma = \frac{2\pi\alpha^2}{\hbar L^6} \delta(E_f - E_i - \hbar\omega) \sum_{jk} \sum_i \frac{e^{-\beta E_i}}{Z} \langle \phi_i | e^{+i\vec{q}\cdot\vec{r_j}} \cdot S_j^- | \phi_f \rangle \langle \phi_f | e^{-i\vec{q}\cdot\vec{r_k}} S_k^+ | \phi_i \rangle$

$\delta(E_f - E_i - \hbar\omega) = \frac{1}{2\pi\hbar} \int dt\, e^{-i\frac{(E_f - E_i)t}{\hbar}} \cdot e^{+i\omega t}$

$\Gamma = \frac{\alpha^2}{\hbar^2 L^6} \sum_{jk} \sum_i \frac{e^{-\beta E_i}}{Z} \langle \phi_i | e^{+\frac{iHt}{\hbar}} S_j^- e^{+i\vec{q}\cdot\vec{r_j}} \cdot e^{-\frac{iHt}{\hbar}} | \phi_f \rangle \langle \phi_f | e^{-i\vec{q}\cdot\vec{r_k}} S_k^+ | \phi_i \rangle$

$\int dt\, e^{+i\omega t}$

$= \frac{\alpha^2}{\hbar L^6} \int dt\, e^{+i\omega t}$ ~~$\langle\langle e^{+i\vec{q}\cdot\vec{r_j}(t)} S_j^-(t) \rangle\rangle$~~ e

$\underbrace{\sum_{j,k} \langle\langle S_j^-(t)\, e^{+i\vec{q}\cdot\vec{r_j}(t)}\, e^{-i\vec{q}\cdot\vec{r_k}} \cdot S_k^+ \rangle\rangle}_{\to S^{-+}(\vec{q}, \omega)}$

E3. problem 2.

(2a) classical $\Pi(q,t) = \langle e^{iqX(t)} e^{-iqX} \rangle = e^{-q^2\langle X_0^2\rangle(1-\cos(\Omega t))}$

$H = \frac{P^2}{2M} + \frac{1}{2}\Omega^2 X^2$ is solved by $X(t) = X_0\cos(\Omega t) + \frac{P_0}{M\Omega}\sin(\Omega t)$

$\Pi(q,t) = \langle e^{iqX(t)} e^{-iqX_0} \rangle = \langle e^{iq(X(t)-X_0)} \rangle$ Ex.1 : $\langle e^{i\alpha x}\rangle = e^{-\frac{1}{2}\alpha^2\langle x^2\rangle}$

$= \frac{1}{Z}\int dp\, e^{-\beta\frac{P^2}{2M}} \int dx\, e^{-\beta\frac{1}{2}M\Omega^2 X^2} \cdot e^{iq(X(t)-X_0)}$

$\langle A(t)\rangle = \langle A(0)\rangle$

$= e^{-\frac{1}{2}q^2\langle (X(t)-X_0)^2\rangle} \rightarrow -\frac{1}{2}q^2\langle X^2(t) - 2X(t)X_0 + X_0^2\rangle = q^2\langle X_0^2 - X(t)X_0\rangle$

$= q^2\left\{\langle X_0^2\rangle(1-\cos(\Omega t)) - \underbrace{\frac{\langle P_0X_0\rangle}{M\Omega}\sin(\Omega t)}_{\to 0}\right)$.

$= e^{q^2\langle X_0^2\rangle(1-\cos(\Omega t))}$

(2.b) $S(q,\omega) = \int dt\, e^{+i\omega t}\,\Pi(q,t) = \int_{-\infty}^{+\infty} dt\, e^{i\omega t} e^{-q^2\langle X_0^2\rangle(1-\cos(\Omega t))}$

$q^2\langle\langle X_0^2\rangle\rangle = \alpha$, $S(q,\omega) = e^{-\alpha}\int dt\, e^{i\omega t}(1 + \alpha\cos(\Omega t) + \frac{\alpha^2}{2}\cos^2(\Omega t) + \cdots + \frac{\alpha^n}{n!}\cos^n\Omega t$

$\cos(\Omega t) = \frac{1}{2}(e^{+i\Omega t} + e^{-i\Omega t}) = e^{-\alpha}\int dt\, e^{+i\omega t}(1 + \frac{\alpha}{2}(e^{i\Omega t} + e^{-i\Omega t}) + \cdots + \frac{\alpha^n}{2^n n!}(e^{+i\Omega t} + e^{-i\Omega t})^n + \cdots$

$= e^{-\alpha}\int dt\, e^{+i\omega t} \sum_{n=0}^{+\infty}\left(\frac{\alpha}{2}\right)^n \frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k} e^{-i(n-2k)\Omega t}$

$= \cancel{e^{-\alpha}\sum_{k=0}^{n}}\; e^{-\alpha}\sum_{n=0}^{+\infty}\left(\frac{\alpha}{2}\right)^n \sum_{k=0}^{n}\frac{1}{k!(n-k)!}\cdot 2\pi\,\delta(\omega - (n-2k)\Omega)$.

(2.c) $g(t) = \langle \hat{X}^2(t) \rangle (\cos(\Omega t) - 1) - i x_0^2 \sin(\Omega t)$, $x_0 = \sqrt{\frac{\hbar}{2m\Omega}}$

$\hat{x} = x_0 (a^\dagger + a)$, $\hat{p} = \frac{i\hbar}{2x_0}(a^\dagger - a)$, $\Pi(q,t) = \langle e^{iq\hat{x}(t)} e^{-iq\hat{x}(0)} \rangle = ?$

$$\hat{X}(t) = e^{+\frac{iHt}{\hbar}} x_0 (a^\dagger + a) e^{-\frac{iHt}{\hbar}} = x_0 \{a^\dagger(t) + a(t)\}$$

$$\dot{a}^\dagger(t) = \frac{i}{\hbar}[\hat{H}, a^\dagger] = i\Omega[a^\dagger a, a^\dagger] = i\Omega a^\dagger [a, a^\dagger] = i\Omega a^\dagger \Rightarrow a^\dagger(t) = e^{+i\Omega t} a^\dagger$$

$$\dot{a}(t) = \frac{i}{\hbar}[\hat{H}, \hat{a}] = -i\Omega [a^\dagger a, a] = -i\Omega a \Rightarrow a(t) = e^{-i\Omega t} a$$

$$\hat{X}(t) = x_0 \{ e^{+i\Omega t} a^\dagger + e^{-i\Omega t} a \}.$$

BCH

$$e^X e^Y = e^Z , \quad Z = X + Y + \frac{1}{2}[X,Y] + \cancel{\frac{1}{12}[X,[X,Y]]} + \cdots , \text{ if } [X,Y] = \text{constant}.$$

$$\frac{1}{2}[\hat{X}(t), \hat{X}] = \frac{1}{2} q^2 x_0^2 \{ e^{-i\Omega t} \overset{+1}{[a, a^\dagger]} + e^{+i\Omega t} \overset{-1}{[a^\dagger, a]} \} = q^2 x_0^2 \{ -i \sin(\Omega t) \}.$$

$$e^{iq\hat{x}(t)} \cdot e^{-iq\hat{x}} = e^{iq(\hat{x}(t) - \hat{x})} \cdot e^{q^2 x_0^2 (i \sin(\Omega t))}$$

$$\langle e^{\bar{u} a^\dagger} e^{ua} \rangle = e^{-|u|^2 \langle n \rangle}$$

$$e^{iq\hat{x}(t)} e^{-iq\hat{x}} = e^{\underbrace{iqx_0(e^{+i\Omega t} - 1)}_{\bar{u}} a^\dagger + \underbrace{iqx_0(e^{-i\Omega t} - 1)}_{-u} a} \, e^{q^2 x_0^2 (-i \sin(\Omega t))}$$

$$= e^{\bar{u} a^\dagger - ua} \cdot \boxed{}$$

$$|u|^2 = q^2 x_0^2 (e^{+i\Omega t} - 1)(e^{-i\Omega t} - 1) = q^2 x_0^2 (2 - 2\cos(\Omega t)).$$

$$e^{\bar{u} a^\dagger - ua} = e^{\bar{u} a^\dagger} e^{-ua} \cdot e^{+\frac{1}{2}|u|^2 [a^\dagger, a]} = e^{\bar{u} a^\dagger} e^{-ua} e^{q^2 x_0^2 \{\cos(\Omega t) - 1\} \cancel{(1 - \cos(\Omega t))}}$$

$$\langle e^{\bar{u} a^\dagger} e^{-ua} \rangle = e^{-|u|^2 \langle n \rangle} = e^{q^2 x_0^2 (\cos(\Omega t) - 1) 2 \langle n \rangle}$$

$$\langle e^{iq\hat{x}(t)} \cdot e^{-iq\hat{x}} \rangle = e^{q^2 x_0^2 \{ (\cos(\Omega t) - 1)(2\langle n \rangle + 1) - i \sin(\Omega t) \}}.$$

$$\langle \hat{X}^2(t) \rangle = \langle \hat{X}^2 \rangle = x_0^2 \langle \underbrace{a^\dagger a^\dagger}_{0} + \underbrace{a^\dagger a}_{\hat{n}} + \underbrace{a a^\dagger}_{\hat{n}+1} + \underbrace{aa}_{0} \rangle = x_0^2 \{ 2\langle n \rangle + 1 \}$$