Problem set 12

Problem 1 Landau meets Dirac

In the lecture we discussed the integer quantum Hall effect (IQHE) for a two-dimensional system of free electrons in a perpendicular magnetic field in the Landau gauge. Let us now look at the case where an underlying lattice structure modifies the dispersion relation. We consider the case of a Graphene sheet at half filling and work with a linearized single-particle Hamiltonian describing the electrons in the vicinity of the Fermi energy ε_F

$$\hat{h}_D = v_F(\tau_z\sigma_x\hat{p}_x + \sigma_y\hat{p}_y) \tag{1}$$

with $\tau_z = \pm 1$ describing electrons at the K or K' point of the first Brillouin zone and $\sigma_{x,y}$ are the Pauli x, y matrices (remember that the two components of the wavefunctions describe the amplitudes in the conduction/valence band). We again introduce the magnetic field via the minimal coupling approach, replacing the momentum operator with the canonical momentum $\hat{\vec{p}} \to \hat{\vec{\pi}} = \hat{\vec{p}} + \frac{e}{c}\vec{A}(\hat{\vec{r}})$. Here, the vector potential is chosen such that (in position space) $\vec{B} = \nabla \times \vec{A}$ and $\vec{B} = -B\vec{e}_z$.

Solving the Dirac equation eq. (1) in the presence of the vector potential turns out to be simpler without choosing a specific gauge. Therefore, as a warmup exercise, we solve the same problem for the free electron case:

$$\hat{h} = \frac{\hat{\vec{\pi}}^2}{2m_e} \ . \tag{2}$$

(1.a) Show that $[\hat{\pi}_x, \hat{\pi}_y] = \mathrm{i}\frac{\hbar^2}{l^2}$, with the magnetic length $l = \sqrt{\frac{\hbar c}{eB}}$.

(1.b) From $\hat{\pi}_{x,y}$, construct ladder operators $\hat{a}^{(\dagger)}$ obeying the canonical commutation relations $[\hat{a}, \hat{a}^\dagger] = 1$ and show that eq. (2) can be rewritten as

$$\hat{h} = \hbar\omega_c\left(\hat{n} + \frac{1}{2}\right) \ , \tag{3}$$

with $\hat{n} = \hat{a}^\dagger\hat{a}$.

(1.c) We now turn to the solution of eq. (1). Show that $[\hat{h}_D^2, \hat{n}] = 0$ and thus $\hat{h}_D^2 = \varepsilon_D^2\mathbf{W}(\hat{n})$ where $\varepsilon_D \in \mathbb{R}$ denotes the energy in the nth eigenstate and $\mathbf{W}(\hat{n})$ is a 2×2 diagonal matrix, which only depends on $\hat{n}$. Determine the explicit form of ε_D and show in particular that $n \in \mathbb{Z}$. Does ε_D depend on τ_z?

(1.d) Now we follow the argumentation in the lecture, i.e., that a filled Landau level gives rise to a perfect conductance channel with transverse (Hall) conductance $\sigma_H = \frac{e^2}{\hbar}$. Therefrom, derive the anomalous IQHE sequence for the conductance of Graphene

$$\sigma_H = 4\left(n + \frac{1}{2}\right)\frac{e^2}{\hbar} \ . \tag{4}$$

Explain the appearance of the possibly negative sign in σ_H.

Problem 2 A useful identity

We want to show a useful identity for the Berry curvature, which is defined for a continuous path $\mathcal{C}$ in the parameters $\vec{R} \in \mathbb{R}^k$ via $\omega_{n,\mu\nu}(\vec{R}) = \partial_{R_\mu}\mathcal{A}_{n,\nu} - \partial_{R_\nu}\mathcal{A}_{n,\mu}$ where n denotes the instantaneous eigenstate $|\psi_n(\vec{R})\rangle$ of a single-particle Hamiltonian $\hat{h}(\vec{R})$ generating the adiabatic trajectory through parameter space along $\mathcal{C}$. Furthermore, $\vec{\mathcal{A}}_n$ denotes the Berry connection $\vec{\mathcal{A}}_n = \mathrm{i}\,\langle\psi_n(\vec{R})|\nabla_{\vec{R}}|\psi_n(\vec{R})\rangle$. Show that the Berry curvature can also be written as

$$\omega_{n,\mu,\nu}(\vec{R}) = \mathrm{i}\sum_{n'\neq n}\frac{\langle\psi_n(\vec{R})|\partial_{R_\mu}\hat{h}|\psi_{n'}(\vec{R})\rangle\,\langle\psi_{n'}(\vec{R})|\partial_{R_n u}\hat{h}|\psi_n(\vec{R})\rangle - \mathrm{h.c.}}{(\varepsilon_n(\vec{R}) - \varepsilon_{n'}(\vec{R}))^2}\,, \tag{5}$$

where $\varepsilon_n(\vec{R})$ denote the eigenvalues of $\hat{h}(\vec{R}$ belonging to the instantaneous eigenstates $|\psi_n(\vec{R})\rangle$.