Problem set 11

Problem 1 A clean, narrow conductor

We consider a mesoscopic, narrow wire connected to two reservoirs and try to determine its conductance properties. The wire is supposed to be disorder free and electrons moving inside the wire scatter elastically, only. We formulate the isolated single-particle problem assuming that it is described by a Hamiltonian

$$\hat{h} = \hat{h}_0 + \hat{V} \tag{1}$$

where $\hat{h}_0$ denotes a usual, periodic single-particle lattice Hamiltonian with single-band dispersion relation $\varepsilon_{\vec{k}} = \frac{\hbar^2 \vec{k}^2}{2m^*}$ and $\hat{V}$ is a potential modelling hard walls, i.e., in position space we have $V = 0$ if $\vec{r} \in \Omega$ and $V = \infty$ otherwise, where Ω denotes the spatial region of the wire.

(1.a) We treat the wire as a two-dimensional device with length $L_x = L$, width $L_y = W$, $W \ll L$ and assume that $k_F \cdot W \approx 1$, where k_F denotes the Fermi momentum along the x-direction. Show that, up to the oscillating Bloch prefactor, the eigenstates (in position space) and eigenvalues of this Hamiltonian are given by

$$\psi_{nk}(x,y) = \sqrt{\frac{2}{LW}} \mathrm{e}^{\mathrm{i}kx} \sin(\frac{n\pi y}{W}) \ , \quad \varepsilon_{nk} = \frac{\hbar^2 k^2}{2m^*} + \frac{\hbar^2 \pi^2}{2m^* W^2} n^2 \tag{2}$$

Here, k denotes the momentum along the x-direction and $n \in \mathbb{N}$ (*Hint: Make explicit use of the specific shape of the wire.*)

(1.b) Now we take into account the electron reservoirs. We assume that a left reservoir is (adiabatically) connected to the wire at $x = 0$ and a right reservoir at $x = L$. The reservoirs should be much larger than the wire such that the chemical potentials μ_L/μ_R of the left/right reservoirs are well defined and $\mu_L > \mu_R$. We furthermore assume that there are no reflections (no impurities in the wire!) and thus in the steady state the distribution functions for right moving electrons ($k > 0$) is given by $f^0(\varepsilon_{nk} - \mu_L)$ and the one for left moving electrons ($k < 0$) is given by $f^0(\varepsilon_{nk} - \mu_R)$ (here, $f^0(\varepsilon_{nk} - \mu)$ denotes the Fermi-Dirac distribution function). Show that the netcurrent for electrons (spin up and spin down) is given by

$$I = -\frac{e}{\pi} \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}k \, v_{nk} \left(f^0(\varepsilon_{nk} - \mu_L)\theta(k) + f^0(\varepsilon_{nk} - \mu_R)\theta(-k) \right) \ , \tag{3}$$

where v_{nk} denotes the electron velocity and $\theta(x)$ is the step function.

(1.c) Show that in the limit $T \to 0$, the current reduces to

$$I = -\frac{2e}{\hbar} \sum_{n=1}^{\infty} \int_{\gamma n^2}^{\infty} \mathrm{d}\varepsilon \left[\theta(\mu_L - \varepsilon) - \theta(\mu_R - \varepsilon) \right] \ , \tag{4}$$

with $\gamma = \hbar^2 \pi^2 / 2m^* W^2$.

(1.d) The voltage drop along the wire is simply given by the differrence between the reservoirs chemical potentials: $-eV = \mu_L - \mu_R$. Assume that we are working in the linear response regime, i.e., $\mu_L - \mu_R$ is small such that the quantity defined by $N(\mu) = \sum_{n=1}^{\infty} \theta(\mu - \gamma n^2)$ satisfies $N(\mu_L) \approx N(\mu_R)$ (N is called the number of *open channels*). Show using that assumptions, the conductance G, defined by $I = GV$, of the wire is given by

$$G = N(\mu)\frac{2e^2}{\hbar} , \tag{5}$$

where we set $\mu_L \equiv \mu_R := \mu$. Sketch the conductance as a function of the voltage drop and explain why the specific dependency on V is not influencing the properties of "normal" wires in everydays electronics (for instance your electronic water kettle).