Problem set 11

(1.a) Eigen function of $\hat{h}_0$ satisfies

$$h_0 |\psi_{\vec{k}}\rangle = \varepsilon_{\vec{k}} |\psi_{\vec{k}}\rangle$$

Where $\varepsilon_{\vec{k}} = \frac{\hbar^2 \vec{k}^2}{2m^*}$. Since $|\psi_{\vec{k}}\rangle$ is Bloch wave,

$$|\psi_{\vec{k}}\rangle = e^{i\vec{k}\cdot\vec{x}} u_{\vec{k}} = e^{ik_x x} e^{ik_y y} u_{k_x, k_y}(x,y)$$

Obviously, $\varepsilon_{k_x, k_y} = \varepsilon_{k_x, -k_y}$, superimposing their eigen states and assuming $u_{k_x, k_y} = u_{k_x, -k_y}$.

$$|\phi\rangle = \alpha |\psi_{k_x, k_y}\rangle + \beta |\psi_{k_x, -k_y}\rangle = (\alpha e^{ik_y y} + \beta e^{-ik_y y}) e^{ik_x x} u_{k_x, k_y}(x,y)$$

At the boundary of infinite deep potential well, stationary waves satisfy boundary condition

$$\begin{cases} \phi(x,0) = (\alpha+\beta) e^{ik_x x} u_{\vec{k}}(x,0) = 0 & \mapsto (\alpha e^{ik_y y} + \beta e^{-ik_y y}) \propto \sin(k_y y) \\ \phi(x,W) = (\alpha e^{ik_y W} + \beta e^{-ik_y W}) e^{ik_x x} u_{\vec{k}}(x,W) = 0 & \mapsto \sin(k_y W) = 0 \end{cases}$$

Therefore, the allowed k_ys are $k_y W = n\pi$, simplify $k_x \to k$.

$$|\phi\rangle = e^{ikx} \sin\left(\frac{n\pi y}{W}\right) u \equiv |\psi_{nk}\rangle$$

$\sin\left(\frac{n\pi y}{W}\right)$ gives normalization factor of $\frac{\sqrt{2}}{\sqrt{W}}$, by convention e^{ikx} gives $\frac{1}{\sqrt{L}}$

$$|\psi_{nk}\rangle = \sqrt{\frac{2}{LW}} e^{ikx} \sin\left(\frac{n\pi y}{W}\right) u$$

Corresponding eigenvalue is

$$\varepsilon_{nk} = \frac{\hbar^2 k^2}{2m^*} + \frac{\hbar^2 \pi^2}{2m^* W^2} n^2$$

(1.b) Consider two contacts L, R sitting at $x=0$ and $x=L$ shooting electrons against each other through a "channel" whose electronic structure is characterized by (1.a)

The net current is $I = \sum_n I_n$

Where n is an open channel that can allow electron to scatter through

$$I_n = \left(-\frac{e}{L}\right)\left(\frac{L}{2\pi}\right)\int dk\, v_{nk} f(k)$$

where $\left(-\frac{e}{L}\right)$ is the charge density, $\left(\frac{L}{2\pi}\right)$ is normalization factor of integral over k,

$v_{nk} = \frac{1}{\hbar}\frac{\partial \varepsilon_n}{\partial k}$ is the group velocity of Bloch electrons.

Electrons with $(+k)$ move from L to R, thus carry the filling factor f of contact L

And similarly, $(-k)$ move from R to L, and carry f of contact R

Therefore, $f(k) = \begin{cases} f_L(\varepsilon_k) = f(\varepsilon_k - \mu_L), & k>0 \\ f_R(\varepsilon_k) = f(\varepsilon_k - \mu_R), & k<0 \end{cases}$

Considering spin, an additional $2s$ factor appears

$$I_n = 2s\left(-\frac{e}{2\pi}\right)\left[\int_0^{+\infty} dk\, v_{nk} f(\varepsilon_{nk}-\mu_L) + \int_{-\infty}^{0} dk\, v_{nk} f(\varepsilon_{nk}-\mu_R)\right]$$

$$= 2s\left(-\frac{e}{2\pi}\right)\int dk\, v_{nk} f(\varepsilon_{nk}-\mu_L)\theta(k) + v_{nk} f(\varepsilon_{nk}-\mu_R)\theta(-k)$$

$$I = \sum_n I_n = \left(-\frac{e}{\pi}\right)\int dk\, v_{nk} f(\varepsilon_{nk}-\mu_L)\theta(k) + v_{nk} f(\varepsilon_{nk}-\mu_R)\theta(-k)$$

(1.c) Using $v_{nk} = \frac{1}{\hbar}\frac{\partial \varepsilon_n}{\partial k}$

$$\int_0^{+\infty} dk\, v_{nk} = \frac{1}{\hbar}\int_0^{+\infty} dk \frac{\partial \varepsilon_n}{\partial k} = \frac{1}{\hbar}\int_{\varepsilon_0}^{\varepsilon_{max}\ (+\infty)} d\varepsilon_n$$

$$\int_{-\infty}^{0} dk\, v_{nk} = \frac{1}{\hbar}\int_{-\infty}^{0} dk \frac{\partial \varepsilon_n}{\partial k} = \frac{1}{\hbar}\int_{\varepsilon_{max}}^{\varepsilon_0} d\varepsilon_n = -\frac{1}{\hbar}\int_{\varepsilon_0}^{\varepsilon_{max}\ (+\infty)} d\varepsilon_n$$

$$I_n = 2s\cdot\left(-\frac{e}{2\pi}\right)\cdot\frac{1}{\hbar}\int_{\varepsilon_0}^{+\infty} d\varepsilon_n f(\varepsilon_n-\mu_L) - f(\varepsilon_n-\mu_R)$$

Take $T \to 0$

$$= -\frac{2se}{h}\int_{\varepsilon_{0,n}}^{+\infty} d\varepsilon_n\, \theta(\varepsilon_n+\mu_L) - \theta(\varepsilon_n+\mu_R)$$

$$I = -\frac{2se}{h}\sum_n\int_{\gamma n^2}^{+\infty} d\varepsilon\, (\theta(\mu_L-\varepsilon) - \theta(\mu_R-\varepsilon)).$$

(1.d) When $\varepsilon_{0,n} = \gamma n^2 < \mu_R < \mu_L$, $\int_{\gamma n^2}^{+\infty} d\varepsilon\, (\theta(\mu_L-\varepsilon)-\theta(\mu_R-\varepsilon)) = \mu_L - \mu_R$

When $\varepsilon_{0,n} > \mu_L > \mu_R$, $\int_{\gamma n^2}^{+\infty} d\varepsilon\, (\cdots) = 0$

The voltage drop along the circuit induces

$$-eV = \mu_L - \mu_R$$

The net current is $I = \sum_{n\,open} -\frac{2se}{h}(\mu_L-\mu_R) = \sum_{n\,open} -\frac{2se}{h}\cdot(-eV) = \sum_{n\,open}\frac{2se^2}{h}\cdot V$

$$= GV$$

$G = N(\mu)\frac{2se^2}{h}$, where $N(\mu) = \sum_n \theta(\mu-\gamma n^2)$

In this setting, when one increases μ, everytime μ goes beyond ~~one~~ the bottom of one band ($\varepsilon_{0,n} = \gamma n^2$), a new channel opens for transport and conducance G increases by $\frac{2se^2}{h}$