Back-of-the-Envelope Physics

Winter Term 2023/24

Sheet 9

1. Determine the density of states in momentum space for a free particle in a box of volume $V = L^3$. Use both periodic boundary conditions as well as fixed boundary conditions (where the wave function vanishes at the boundaries of the box) and compare the two methods.

2. Show that for a system s in thermal contact with a reservoir, with total (fluctuating) energy E_s and average energy $E = \langle E_s \rangle$, the energy fluctuation is given by

$$\Delta E^2 \equiv \langle (E_s - E)^2 \rangle = \frac{\partial^2}{\partial \beta^2} \ln Z \tag{1}$$

where Z is the partition function of the canonical ensemble.

Use this formula to compute the relative energy fluctuation $\Delta E/E$ for the case of the ideal gas.

3. Obtain the free energy F of the photon gas directly from the definition $F = -T \ln Z$ by computing the partition function Z explicitly.

4. The Riemann zeta function can be defined as

$$\zeta(n) \equiv \frac{1}{\Gamma(n)} \int_0^\infty dx \, \frac{x^{n-1}}{e^x - 1} \tag{2}$$

Show that

$$\zeta(n) = \sum_{k=1}^{\infty} \frac{1}{k^n} \tag{3}$$

The energy density in a photon gas can be written as

$$\frac{E}{V} = \frac{6}{\pi^2} \zeta(4) \, T^4 \tag{4}$$

Estimate the numerical coefficient in (4) using (3) and compare with the exact result.