# Back-of-the-Envelope Physics

## Winter Term 2023/24

## Sheet 5

1. Consider the quantum mechanics of a particle of mass $m$ in the 1-dimensional potential $V(r) = \beta r$ for $r > 0$ and $V(r) = \infty$ for $r < 0$.

a) Sketch the potential together with the shape of the ground-state wave function.

b) Use dimensional analysis to find an expression for the energy $E_1$ of the ground state.

c) Determine the dependence of the energy eigenvalues $E_n$ on the quantum number $n$ (for large $n$), using the approximation of the potential by an infinite square well of appropriate width.

d) Estimate the asymptotic behaviour of the wave function for large $r$.

2. The Schrödinger equation for the wave function $u(r)$ of problem 1 is

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + \beta r\, u = Eu \tag{1}$$

for $r > 0$, with boundary condition $u(0) = 0$. Obtain the exact solution for the eigenvalues and eigenfunctions. Compare with the approximate results from problem 1.

*Hint:* Write eq. (1) in terms of a dimensionless variable $s$ through a suitable rescaling of $r = \lambda s$. Similarly, introduce dimensionless energy eigenvalues $\varepsilon$. In this way, eq. (1) can be reduced to the form

$$\frac{d^2u}{dz^2} = zu\,, \tag{2}$$

which is solved by the Airy function

$$\mathcal{A}(z) = \frac{1}{\pi}\int_0^\infty \cos\left(\frac{t^3}{3} + tz\right)\, dt \tag{3}$$

3. The tunneling probability $\mathcal{R}$ for the stellar fusion reaction $p+p \to d+e^{+}+\nu_e$ can be written as $\mathcal{R} = \exp(-\sqrt{E_G/E})$, where $E$ is the energy of the $pp$ collision.

Estimate the Gamow energy $E_G$ for this process, by giving a parametric formula and a numerical evaluation.

4. Including the Boltzmann factor, the tunneling probability from problem 3 becomes $\mathcal{R}_B = \exp(-(\sqrt{E_G/E}+E/T))$. The factor $\mathcal{R}_B(E)$ has a peak at $E = \bar{E}$. Determine the position $\bar{E}(T)$ and the width $\Gamma(T)$ of this peak by expanding the exponent to second order around $\bar{E}$.