Back-of-the-Envelope Physics

Winter Term 2023/24

Sheet 13

1. Consider a black-hole binary with circular orbit at a distance R from the earth. The black holes have mass m each and a mutual distance r.

a) Show that the metric perturbation h from the gravitational radiation at the position of the earth can be estimated to be $h \sim r_s^2/rR$. Estimate the maximum of h numerically for $r_s = 100\,\text{km}$ and $R = 10^9$ light years.

b) Show that the total power radiated from a black-hole binary can be written in the form $d\mathcal{E}/dt \sim (v/c)^5\, v^5/G$, where v is the speed of a black hole on its orbit. Estimate the maximum power numerically.

2. Two black holes with mass m orbit each other at distance r. Estimate the time t it takes until the merger occurs.

3. Show that

$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha-\beta}{2}\,\cos\frac{\alpha+\beta}{2} \tag{1}$$

Use this formula to discuss the superposition of two plane waves with phase factor $\alpha = k_1 x - \omega_1 t$ and $\beta = k_2 x - \omega_2 t$, respectively. Assume that k_1 is close to k_2, such that $k_1 - k_2 \ll (k_1 + k_2)/2 \equiv k$, and similarly for ω_i. Describe how this superposition may be interpreted as a toy model of a wave packet and derive expressions for the corresponding phase velocity v_p and group velocity v_g.

4. Consider a lake with bottom at $z = -h$, (undisturbed) surface at $z = 0$, and infinite extent in the x and y directions. Let the velocity potential φ of a water wave be given by

$$\varphi(x,z) = A\cos(kx - \omega t)\,\cosh k(z+h) \tag{2}$$

a) Compute the velocity field $\vec{v}(t,x,y,z)$ and check that it fulfills the condition for incompressible flow.

b) Obtain the amplitude field $\zeta(t,x)$ describing the moving water surface. Assume that $\zeta \ll \lambda = 2\pi/k$.