Inverse problems and machine learning in medical physics

Fundamentals of tomographic imaging

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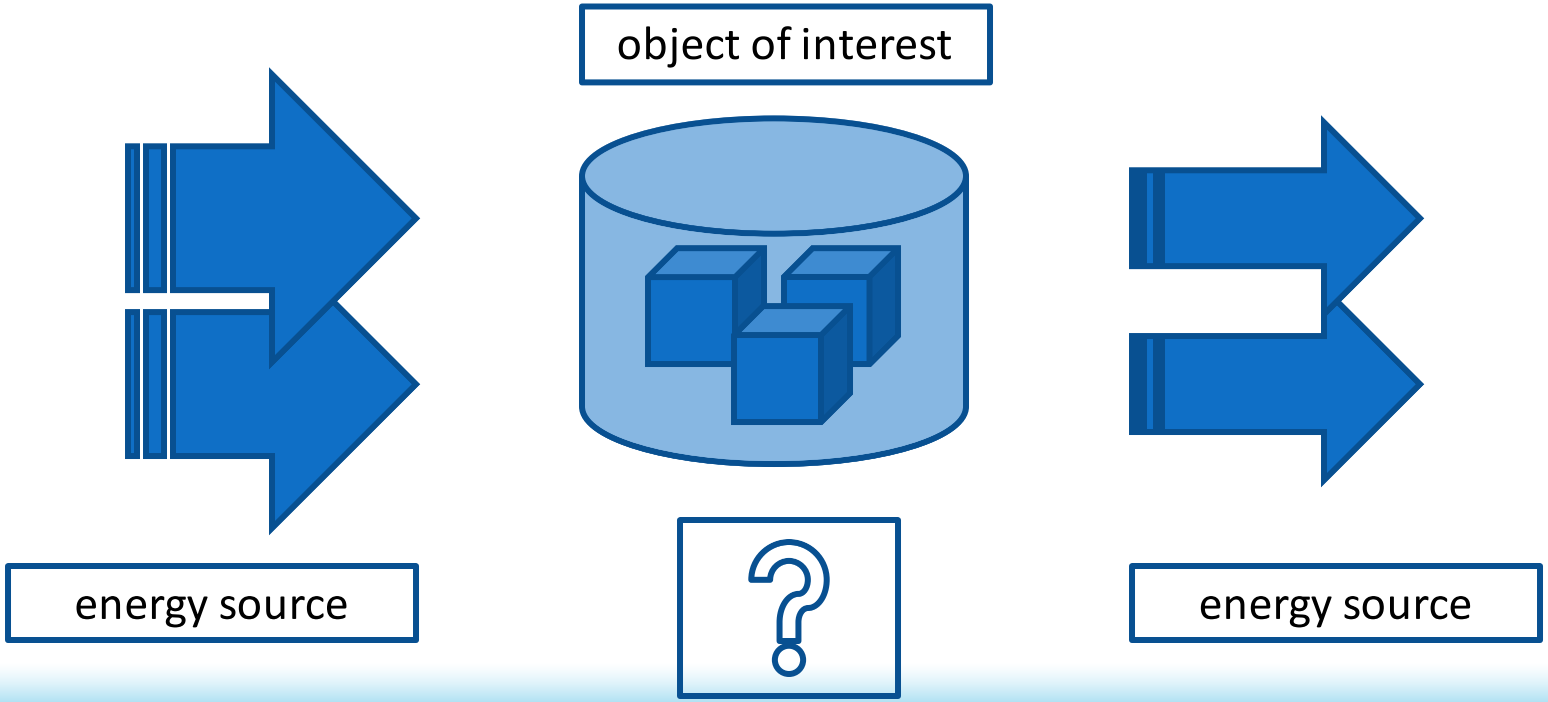
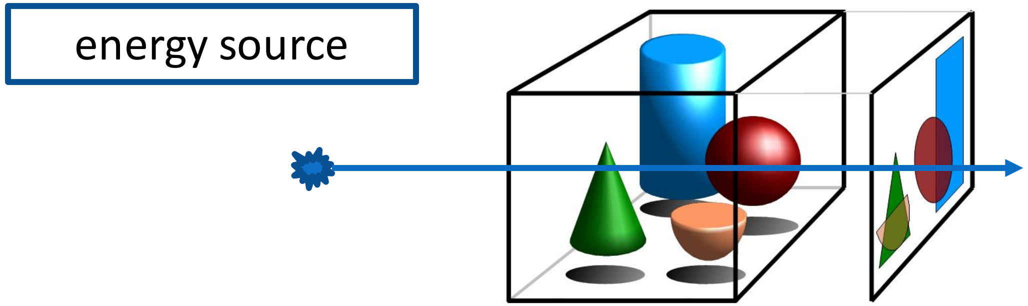
Tomographic imaging

- A tomographic image is a volumetric representation of the physical variables describing the object of interest
- The variables describe the physical properties of the object of interest in terms of the effects on the energy source
  - Depending on the energy source, transmission imaging (external energy source) and emission imaging (energy source inside the object of interest) are defined

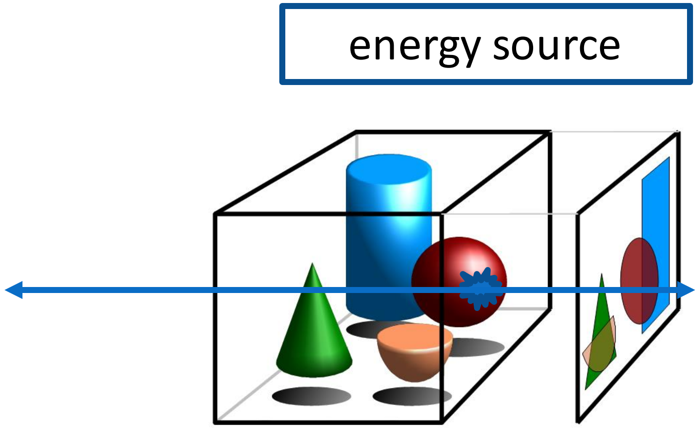


Transmission imaging

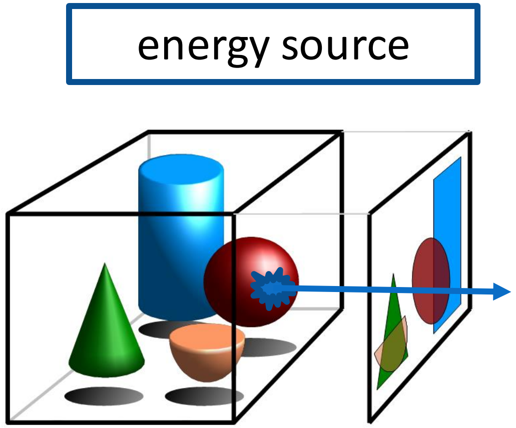


	X-ray Computed Tomography (CT)	ion Computed Tomography (iCT)
Physical properties	Total attenuation	Integral stopping power
Energy sources	Photon beam	Ion beam
Variables	Attenuation coefficients	Stopping power

Emission imaging



Positron emission tomography (PET)



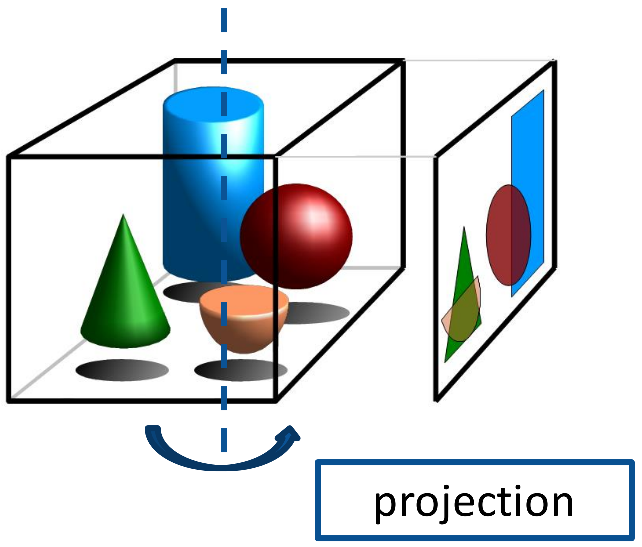
Single photon emission tomography (SPECT)

	PET	SPECT
Physical properties	Emission of annihilation photons	Photon emission
Energy sources	Radioactive nuclei (β^+ emitters)	Radioactive nuclei (γ emitters)
Variables	Emitted counts (time coincidence)	Emitted counts (acceptance angle)

Tomographic imaging

- Tomographic image reconstruction is an inverse problem that aims at finding the cause of the phenomenon
  - Causes of the phenomenon are the physical properties of the object of interest
  - Consequences of the phenomenon are the measured (observed) effects on the energy source
- The measurements are collected in several projections at different projection angles with respect to the rotational axis of the imaging system
- To find out "what is inside" the object of interest is observed from many points of view…

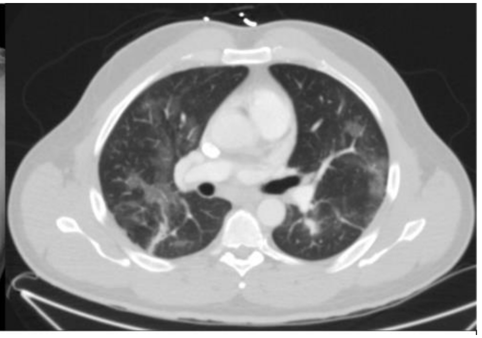
https://en.wikipedia.org/wiki/Tomography



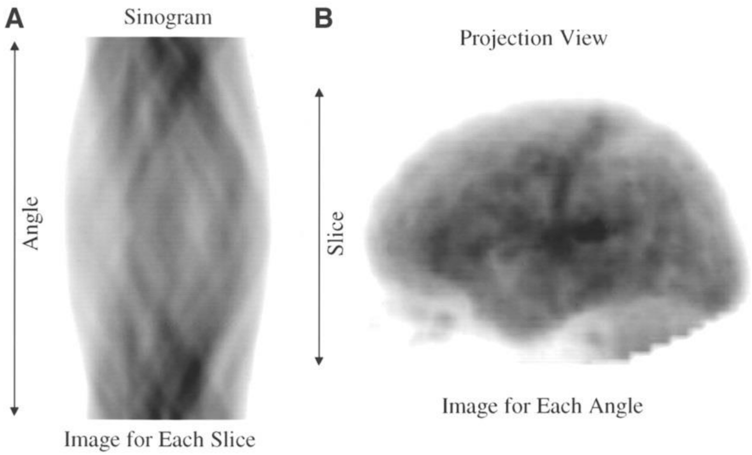
Radiographic and tomographic imaging

- The rotational axis of the imaging system is the axis of the cylindrical scanner (the object of interest is the patient and does not typically rotate)
    - In transmission imaging, the projection is synonymous of radiography
    - In emission imaging, the projection is synonymous of view and the projection is typically visualized as sinogram

radiography



CT



sinogram

view

PET

https://wiki.cancerimagingarchive.net/

Fahey, F. H. (2002). Data acquisition in PET imaging. Journal of nuclear medicine technology, 30(2), 39-49.

Imaging scanners

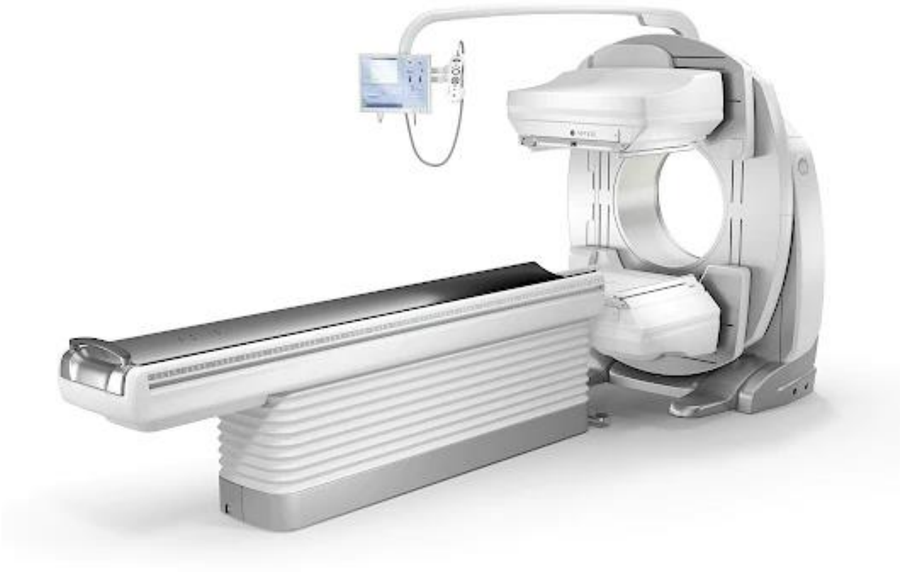
- The acquisition of emission imaging is combined with transmission imaging in modern PET/CT and SCPECT/CT scanners

CT scanner

https://www.siemens-healthineers.com/

PET/CT scanner

https://www.philips.co.in/healthcare/solutions/advanced-molecular-imaging/pet-ct



SPECT/CT scanner

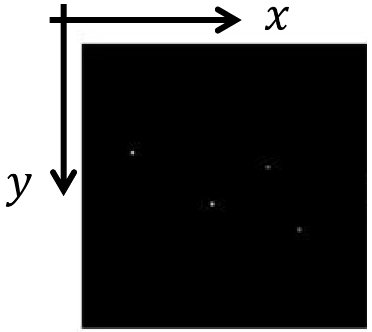
https://www.gehealthcare.com/products/molecular-imaging/nuclear-medicine/

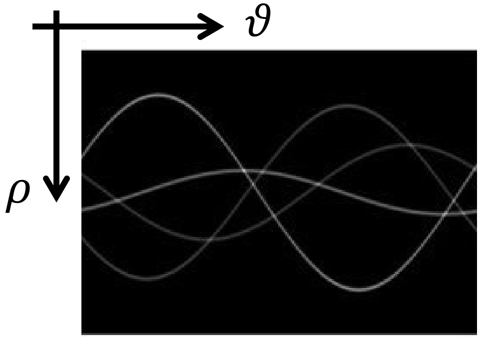
The projection

- The projection is defined as the line integral along l of the function $f(x,y)$ describing the object of interest at a radial distance ρ from the origin

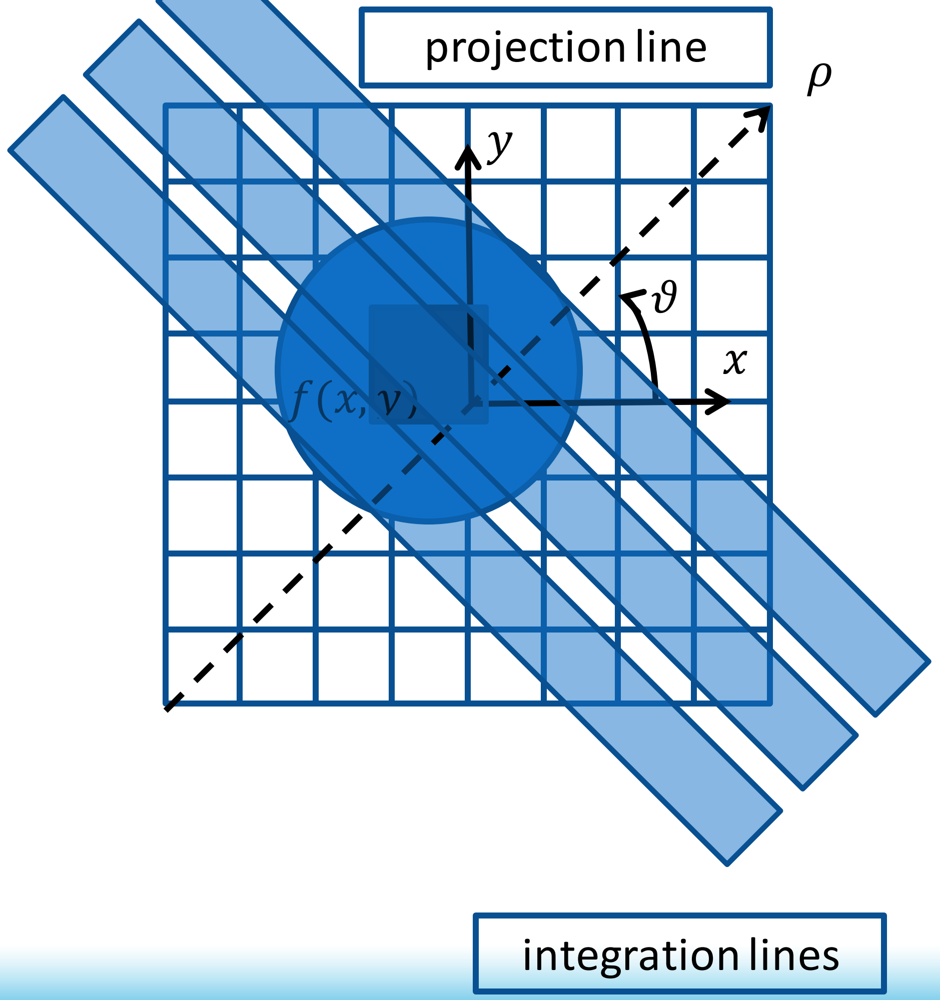
$$p(\rho,\vartheta) = \int_l f(x,y)\,dl$$

- The projection is expressed in polar coordinates (ρ,ϑ)
- The projection of a point in polar coordinates (ρ,ϑ) is a sinusoidal function (i.e., sinogram)





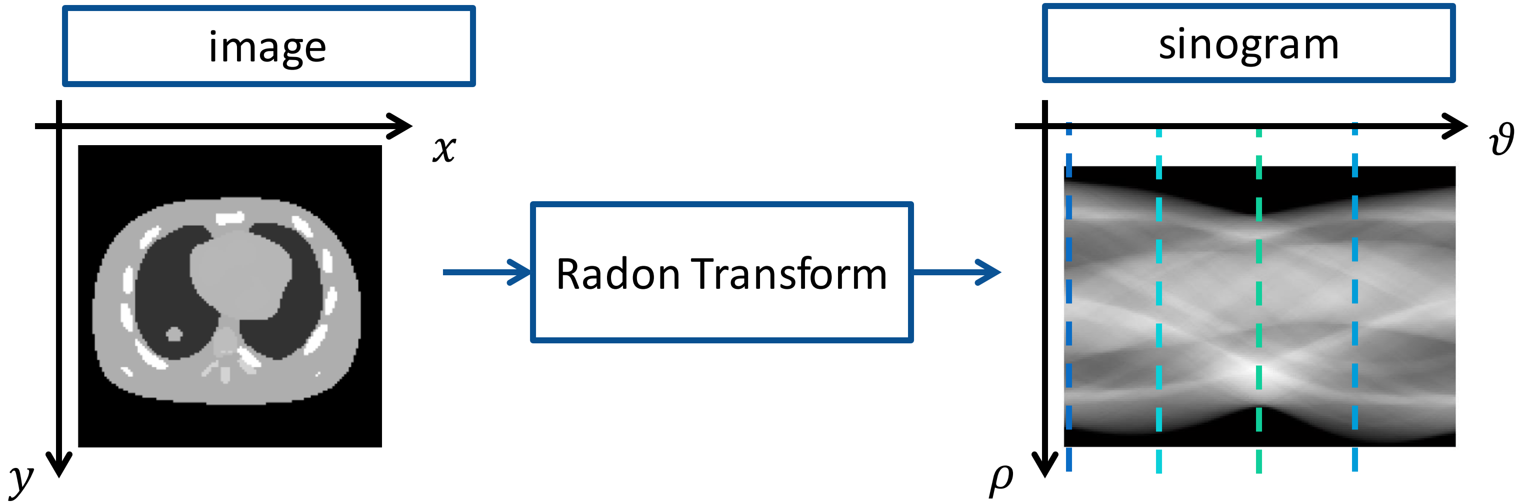
$$\rho = x\cos\vartheta + y\sin\vartheta$$



Tomographic image reconstruction: the Radon Transform

- Tomographic image acquisition can be modelled as a Radon Transform, or sinogram, of the variable describing the physical properties of the object of interest

- The Radon Transform converts an image from spatial domain to sinogram domain, by integrating the variables along the integration lines, as a function of the projection angles



- Tomographic image reconstruction is based on the Radon Transform

Tomographic image reconstruction: the Radon Transform

- The projection as a line integral is converted to an image integral by introducing the Dirac's δ function
    - Dirac's δ function δ(t) is δ(t) = 0 everywhere except in t = 0
- The Radon transform can be written in continuous or discrete forms

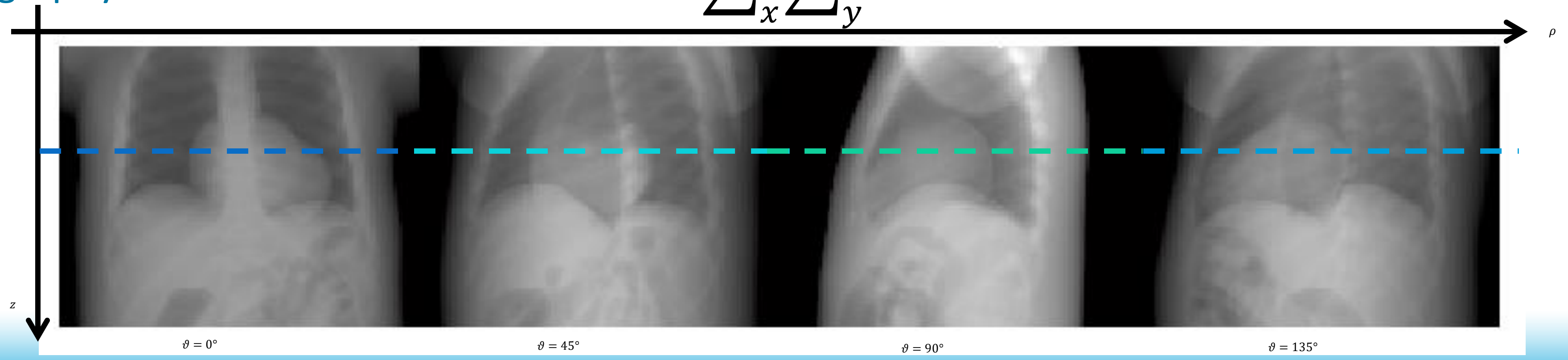
$$p(\rho,\vartheta) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\delta(x cos\vartheta + y sin\vartheta - \rho)dxdy$$

continuous

$$p(\rho,\vartheta) = \sum_x \sum_y f(x,y)\delta(x cos\vartheta + y sin\vartheta - \rho)$$

discrete

- The radiography is written as: $g_\vartheta(\rho,z) = \sum_x \sum_y f(x,y,z)\delta(x cos\vartheta + y sin\vartheta - \rho, z)$



Analytical image reconstruction

- The analytical image reconstruction is based on the Fourier Slice Theorem (or Central Section Theorem)
    - The Fourier Slice Theorem puts in correspondence the 2D Radon Transform with the Fourier Transform (FT) of the 2D image
    - The 2D FT of the image evaluated along the projection line ρ in frequency domain (w_x, w_y) coincides with the 1D FT of the Radon Transform for the same projection line in spatial domain (x, y):

$$\hat{f}_\rho(w_x, w_y) = \int_{-\infty}^{+\infty} R(f)e^{-2\pi i(\rho w_\rho)}d\rho = \hat{R}(w_\rho)$$

^ indicates frequency domain

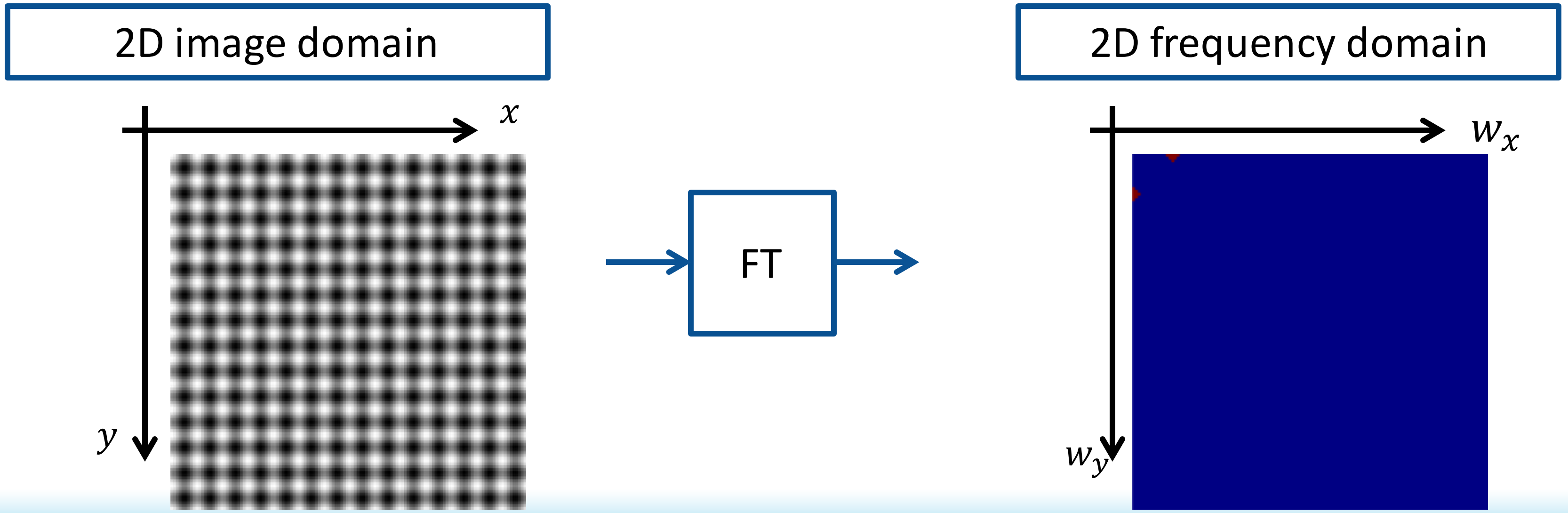
- The analytical image reconstruction is based on the discrete form of Fourier Slice Theorem, according to different implementations

Analytical image reconstruction: the Fourier Transform

- The 2D Fourier Transform (FT) converts an image from 2D spatial domain to 2D frequency domain, by decomposing the image into sine and cosine components (or basis functions)

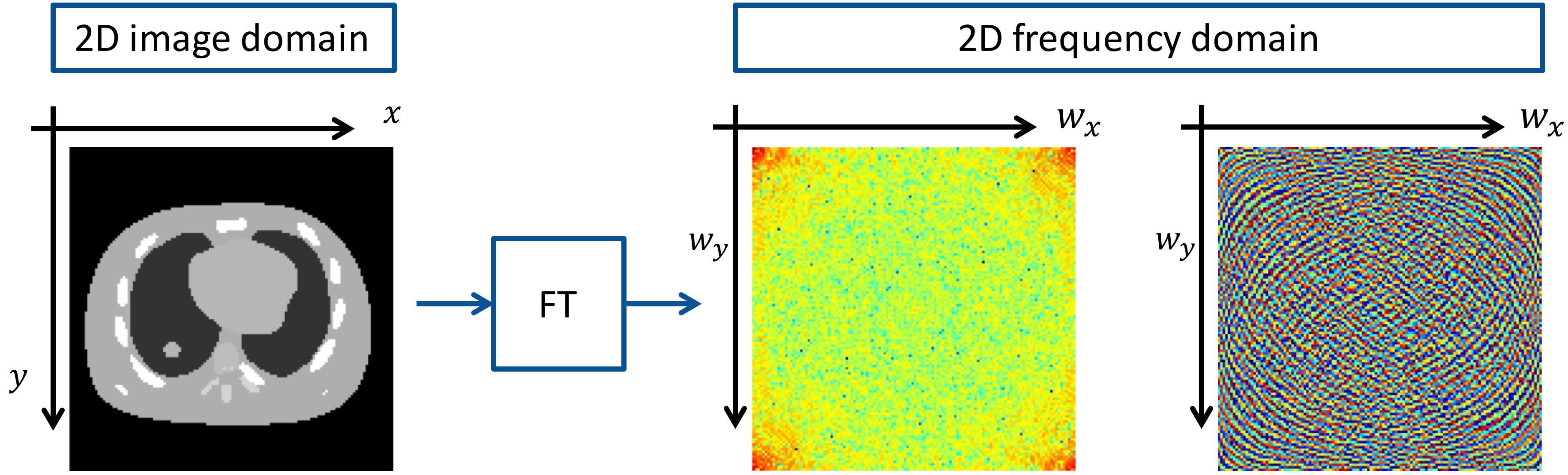
$$\hat{f}(w_x, w_y) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x, y) e^{-2\pi i (x w_x + y w_y)} dx dy$$

  - Two sinusoidal components in spatial domain correspond to two delta components in frequency domain



Analytical image reconstruction: the Fourier Transform

- The 2D FT of an image can be represented as real and imaginary parts
  - The real part represents the amplitude of the sinusoidal components
  - The imaginary part represents the phase of the sinusoidal components



Analytical image reconstruction

- The different algorithms for analytical image reconstruction are derived following these equivalences:

$$f(x,y) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \hat{f}(w_x, w_y) e^{2\pi i (x w_x + y w_y)}\, dw_x dw_y$$

inverse 2D FT

$$f(x,y) = \int_{0}^{\pi}\int_{-\infty}^{+\infty} \hat{f}(w_\rho \cos\vartheta, w_\rho \sin\vartheta) e^{2\pi i w_\rho (x\cos\vartheta + y \sin\vartheta)} |w_\rho| dw_\rho d\vartheta$$

variable substitution

$$f(x,y) = \int_{0}^{\pi}\int_{-\infty}^{+\infty} \hat{f}(w_\rho \cos\vartheta, w_\rho \sin\vartheta) e^{2\pi i w_\rho \rho} |w_\rho| dw_\rho d\vartheta$$

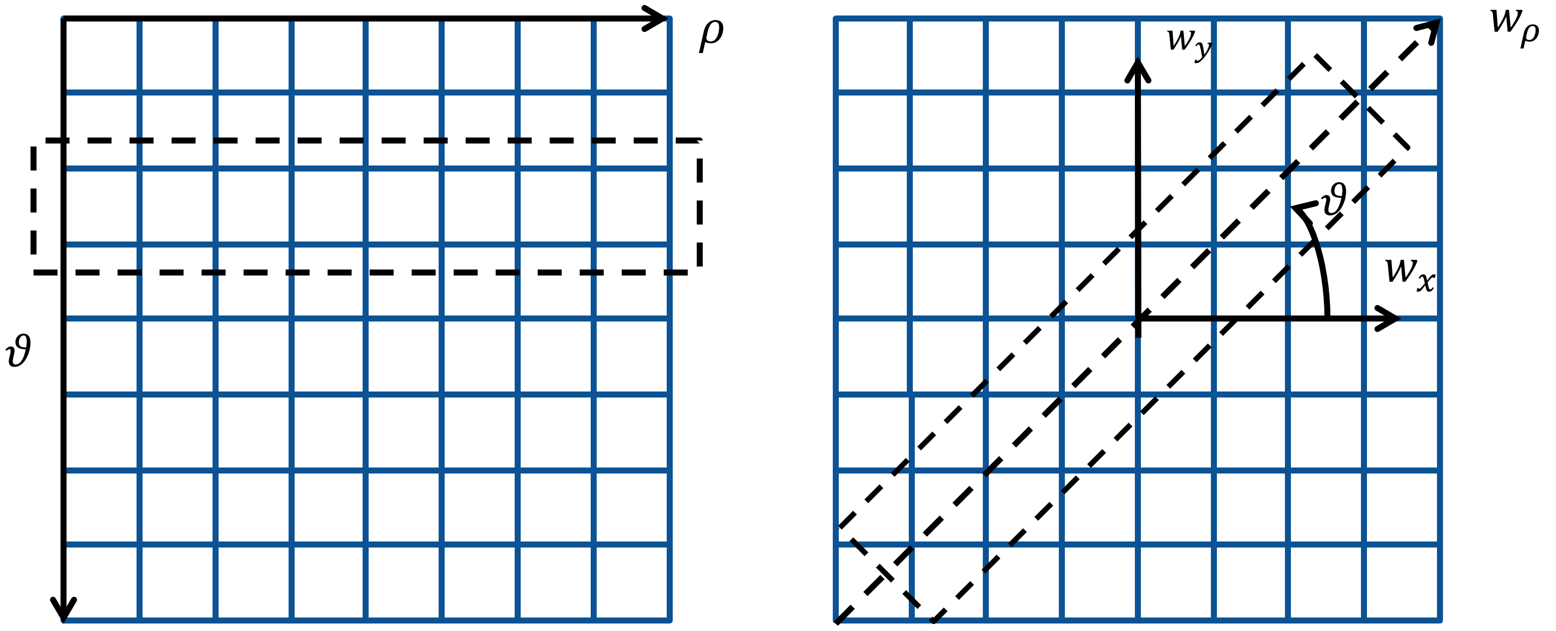
$$f(x,y) = \int_{0}^{\pi}\int_{-\infty}^{+\infty} \hat{R}(w_\rho) e^{2\pi i \rho w_\rho} |w_\rho|\, dw_\rho d\vartheta$$

Fourier Slice Theorem

- The image results as the inverse 2D FT of the 1D FT of the Radon Transform filtered by an high-pass filter (Ramp filter) along each projection line in frequency domain
  - The Ramp filter (high frequencies amplification) derives from the Jacobian determinant of the variable substitution, from Cartesian coordinates to semi-polar coordinates

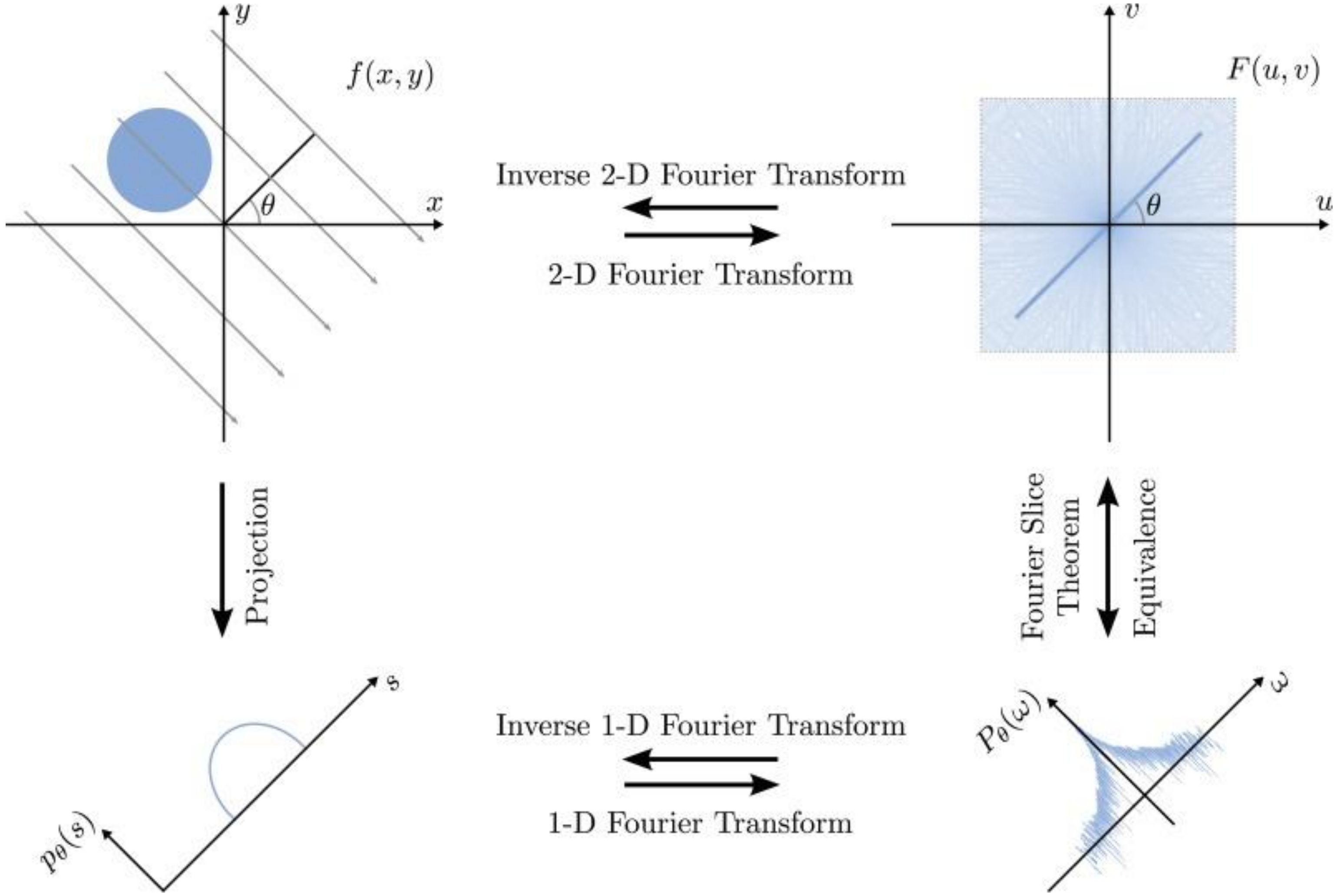
Analytical image reconstruction

- The algorithm for Fourier reconstruction consists in the 1D Fourier transform of the Radon Transform, filtered by an high pass filter (Ramp filter) and interpolated in frequency domain, followed by inverse 2D Fourier transform
  - The algorithm suffers from approximations in filter discretization and interpolation in frequency domain



1D FT → **Ramp filter** as multiplication in frequency domain (1D) → **Interpolation** from polar to Cartesian coordinates → Inverse 2D FT

Analytical image reconstruction



Taubmann O, Berger M, Bögel M, et al. Computed Tomography. 2018 Aug 3. In: Maier A, Steidl S, Christlein V, et al., editors. Medical Imaging Systems: An Introductory Guide [Internet]. Cham (CH): Springer; 2018. Chapter 8. Available from: https://www.ncbi.nlm.nih.gov/books/NBK546157/ doi: 10.1007/978-3-319-96520-8_8

Analytical image reconstruction

- The algorithms for filtered back-projection and convolution back-projection are derived by continuing the previous equivalence as:

$$f(x,y) = \int_0^{\pi} \int_{-\infty}^{+\infty} \hat{R}(w_\rho) e^{2\pi i \rho\, w_\rho} |w_\rho|\, dw_\rho d\vartheta$$

$$f(x,y) = \int_0^{\pi} \int_{-\infty}^{+\infty} \hat{R}(w_\rho) e^{2\pi i \rho\, w_\rho} dw_\rho |w_\rho|\, d\vartheta$$

inverse 1D FT

$$f(x,y) = \int_0^{\pi} R(w_\rho) |w_\rho|\, d\vartheta$$

filtered-back-projection (frequency domain)

$$f(x,y) = \int_0^{\pi} g(\rho,\vartheta) * k_{ramp}(\rho)\, d\vartheta$$

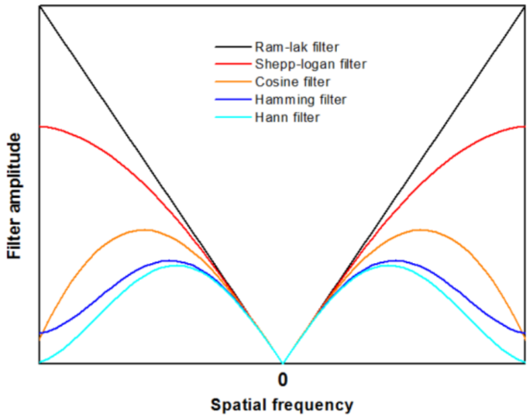
convolution back-projection (spatial domain)

- A multiplication in frequency domain is equivalent to a convolution in spatial domain

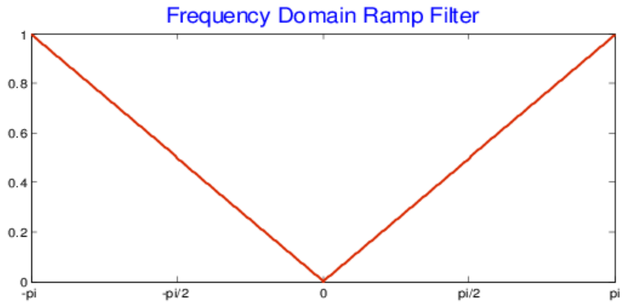
Analytical image reconstruction

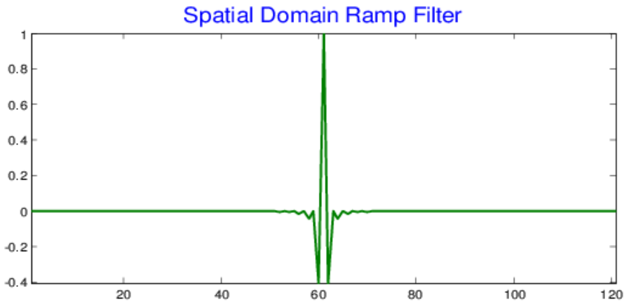
- The image results as the back-projection of the 1D FT of the Radon Transform, filtered in frequency domain by the Ramp filter (filtered back-projection)

- The image results as the back-projection of the Radon Transform, filtered in spatial domain by an high pass filter (convolution back-projection)
  - The Ramp filter is typically weighted/windowed towards the high frequencies to mitigate the noise on the reconstructed image
    - Fundamental trade-off between noise and spatial resolution in imaging!



Lee, S. W., Lee, C. L., Cho, H. M., Park, H. S., Kim, D. H., Choi, Y. N., & Kim, H. J. (2011). Effects of reconstruction parameters on image noise and spatial resolution in cone-beam computed tomography. Journal of the Korean Physical Society, 59(4), 2825-2832.

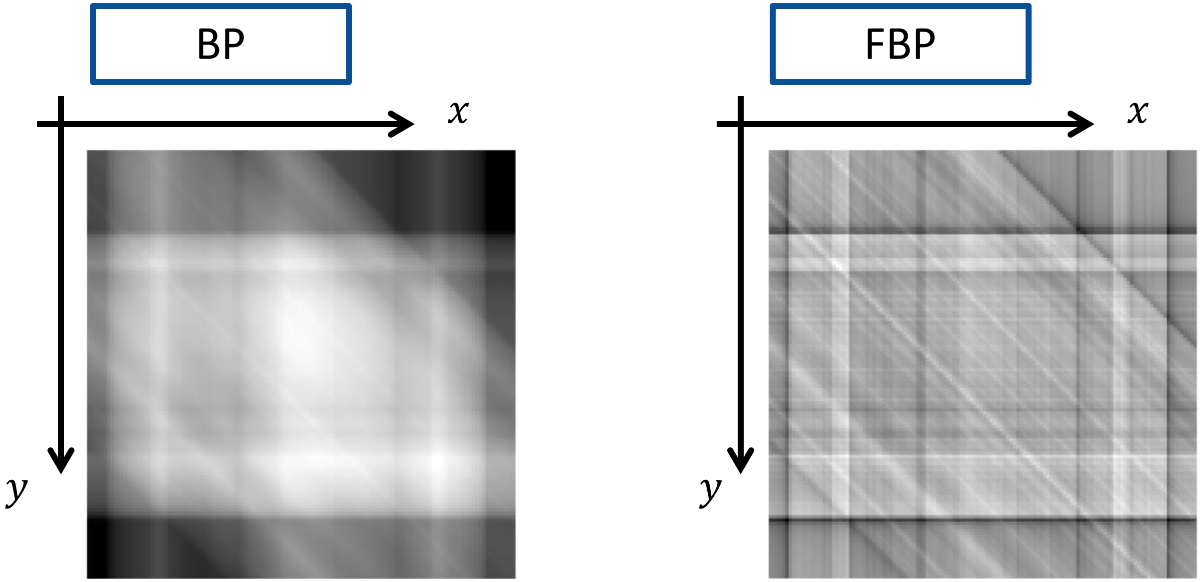




Aggarwal, P., & Mehra, R. (2011). High speed CT image reconstruction using FPGA. International Journal of Computer Applications, 22(4), 7-10.

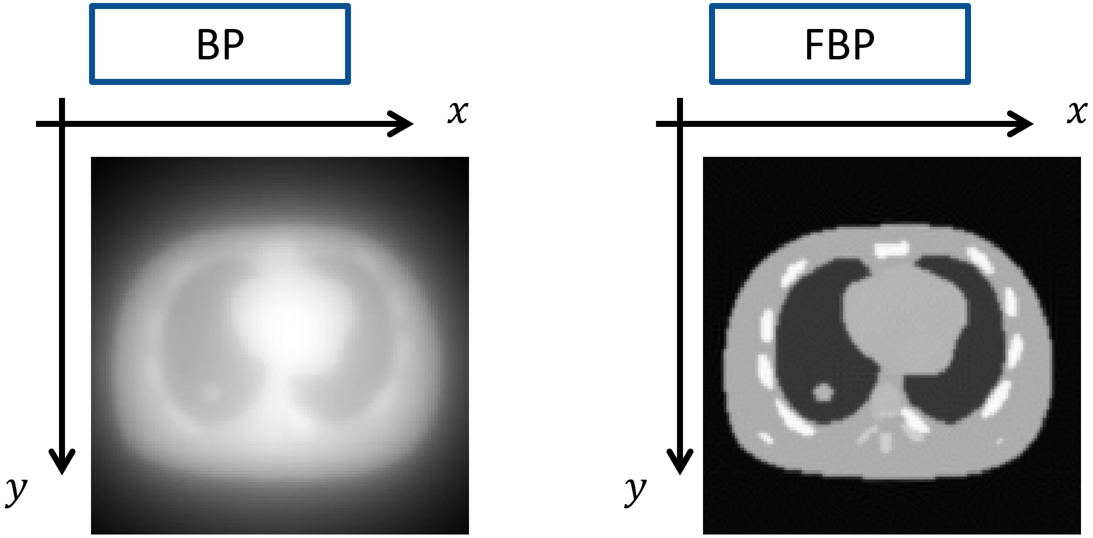
Analytical image reconstruction

- Back Projection (BP) and Filtered Back Projection (FBP) of the projections at angles $\vartheta = 0°$, $\vartheta = 45°$ and $\vartheta = 90°$, number of integration lines $n\rho = 128$ (equal to the number of rows and columns of the image)



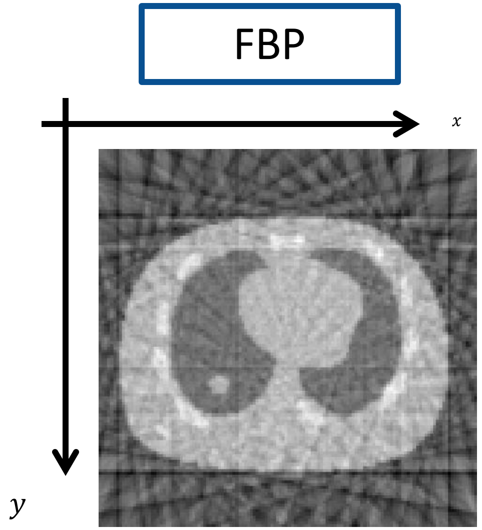
Analytical image reconstruction

- Image reconstructed according to Back Projection (BP) and Filtered Back Projection (FBP) by setting the number of projection angles $n\vartheta = 180$ with spacing $\vartheta = 1°$ and the number of integration lines $n\rho = 128$



Analytical image reconstruction

- Image reconstructed according to Filtered Back Projection (FBP) by setting the number of integration lines $n\rho = 128$ and the number of projection angles $n\vartheta = 18$ with spacing $\vartheta = 10°$



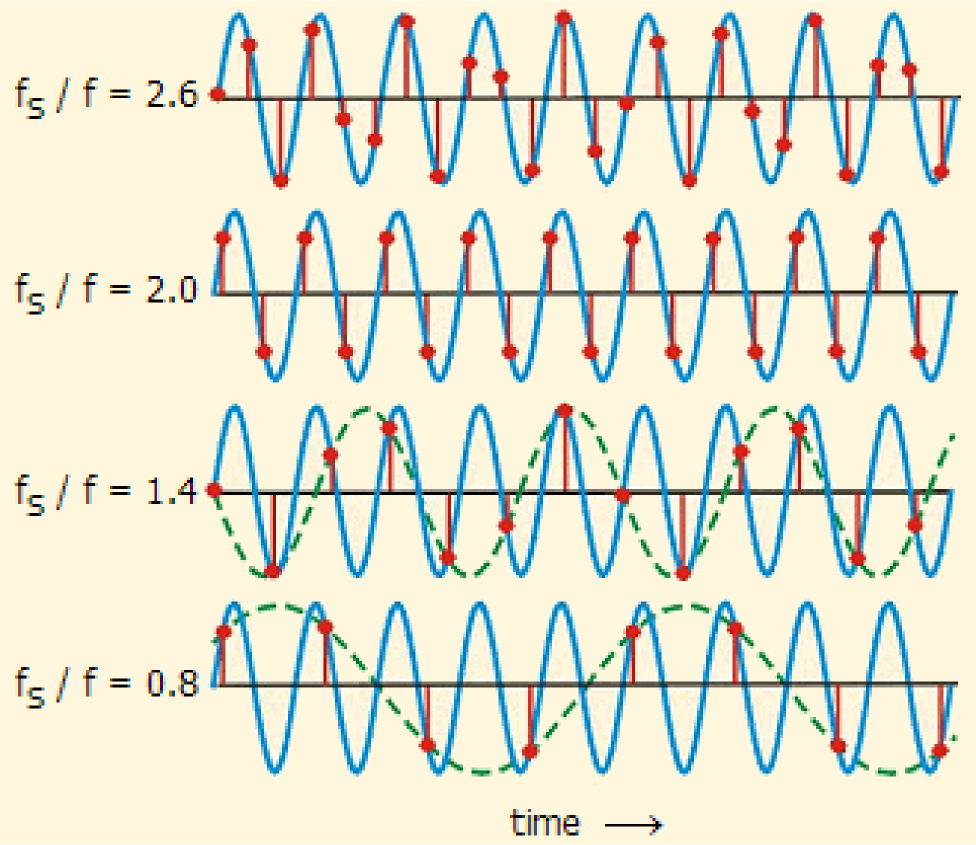
Analytical image reconstruction

- The discrete form of the Fourier Slice Theorem relies on the Nyquist theorem of sampling
  - The Nyquist theorem establishes a sufficient condition on the sampling frequency f_s for capturing (sampling) all the information of the continuous image up to the frequency f
  - The *fs* that guarantees the sufficient condition is: $f_s = 2f$
  - In other words, as the faster variation of the image in frequency domain requires at least 2 samples to be caught, the smaller variation in spatial domain is caught by at least 2 samples (two pixels!)
- The Nyquist theorem of sampling is therefore satisfied for: $\Delta\vartheta = arctan\left(\frac{1}{\frac{\sqrt{N}}{2}}\right)$

  where N is the number of pixels of the image
- An analytical image reconstruction that violates this sufficient condition generates “streaks artifacts” (or star-artifacts) in the 2D image

Analytical image reconstruction

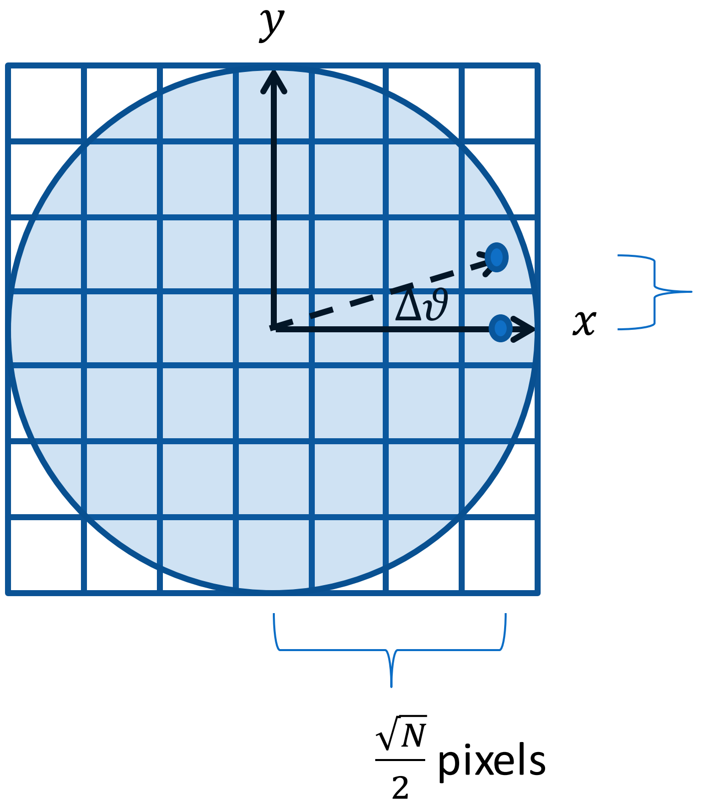
- Intuitive explanation of the Nyquist theorem of sampling for temporal signals



http://195.134.76.37/applets/AppletNyquist/Appl_Nyquist2.html

Analytical image reconstruction

- Intuitive explanation of the Nyquist theorem of sampling for spatial 2D signals (images)
    - The smallest angle able to catch the smallest variation (2 pixels) within the field of view (inscribed circle)



$$\Delta\vartheta = arctan\left(\frac{1}{\frac{\sqrt{N}}{2}}\right)$$

Outlook

- Analytical image reconstruction is based on the continuous form of the Radon Transform
- The Fourier Slice Theorem, provided with the Nyquist theorem of sampling, enables the implementation and application of analytical reconstruction algorithms
    - The hypothesis of continuity for the discrete 2D image and the 2D sinogram can be hardly verified in presence of geometrical constraints (i.e., geometry of the projection lines, angular coverage and angular sampling) and dosimetric constraints (i.e., noise)
- The imaging trade-off between noise and spatial resolution is controlled by the weighting/windowing of the Ramp filter