Problem 1:

3+1 dim, $\mathcal{L} = \frac{f_\pi^2}{4} \mathrm{Tr}\left((\partial_\mu U)^\dagger (\partial^\mu U)\right)$

$$U(x) = \exp\left(\frac{i}{f_\pi} \pi^a(x) \sigma^a\right)$$

(1)

- The Lagrangian is invariant under the global transformation $U \longmapsto L U R$
  with $L \in SU(2)_L$, $R \in SU(2)_R$,
  and $U \longmapsto e^{i\alpha} U$.

→ The global symmetry is $SU(2)_L \times SU(2)_R \times U(1)$

(2)

- $$U(x) = \exp\left(\frac{i}{f_\pi} \pi^a \sigma^a\right) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{i}{f_\pi} \pi^a \sigma^a\right)^n$$
  $$= \sum_{n=0}^{\infty} \frac{1}{(2n)!} \left(\frac{i}{f_\pi} \pi^a \sigma^a\right)^{2n} + \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \left(\frac{i}{f_\pi} \pi^a \sigma^a\right)^{2n+1}$$

- $$\pi^a \sigma^a \pi^b \sigma^b = \pi^a \pi^b \left(\delta_{ab} + i \cancel{\varepsilon_{abc} \sigma^c}\right) = \pi^a \pi^a$$

$$U(x) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} (-1)^n \left(\frac{\pi^a \pi^a}{f_\pi^2}\right)^n + \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} (-1)^n \left(\frac{\pi^a \pi^a}{f_\pi^2}\right)^n \frac{i}{f_\pi} \pi^b \sigma^b$$

$$= \sum_{n=0}^{\infty} \frac{1}{(2n)!} (-1)^n \left(\frac{\sqrt{\pi^a \pi^a}}{f_\pi}\right)^{2n} + \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} (-1)^n \left(\frac{\sqrt{\pi^a \pi^a}}{f_\pi}\right)^{2n+1} i \frac{\pi^b \sigma^b}{\sqrt{\pi^a \pi^a}}$$

$\pi := \sqrt{\pi^a \pi^a}$

$$= \cos\left(\frac{\pi}{f_\pi}\right) \mathbb{1} + i \sin\left(\frac{\pi}{f_\pi}\right) \frac{\pi^a}{\pi} \sigma^a$$

- $$\partial_\mu U = -\sin\left(\frac{\pi}{f_\pi}\right) \frac{1}{f_\pi} \partial_\mu \pi + i \cos\left(\frac{\pi}{f_\pi}\right) \frac{\pi^a}{\pi} \sigma^a \frac{1}{f_\pi} \partial_\mu \pi$$

$$+ i\sin\left(\frac{\pi}{f_\pi}\right)\frac{1}{\pi}(\partial_\mu \pi^a)\sigma^a - i\sin\left(\frac{\pi}{f_\pi}\right)\frac{\pi^a}{\pi^2}\sigma^a\,\partial_\mu\pi$$

$$= -\frac{\pi}{f_\pi^2}\partial_\mu\pi + i\left(\cancel{1} - \frac{1}{2}\frac{\pi^2}{f_\pi^2}\right)\frac{\pi^a}{\pi}\sigma^a\frac{1}{f_\pi}\partial_\mu\pi$$

$$+ i\left(\frac{\pi}{f_\pi} - \frac{1}{6}\frac{\pi^3}{f_\pi^3}\right)\frac{1}{\pi}(\partial_\mu\pi^a)\sigma^a$$

$$- i\left(\cancel{\frac{\pi}{f_\pi}} - \frac{1}{6}\frac{\pi^3}{f_\pi^3}\right)\frac{\pi^a}{\pi^2}\sigma^a\,\partial_\mu\pi + \mathcal{O}\left(\frac{1}{f_\pi^4}\right)$$

$$= -\frac{\pi}{f_\pi^2}\partial_\mu\pi - \frac{i}{2}\frac{\pi}{f_\pi^3}\pi^a\sigma^a\partial_\mu\pi + i\frac{1}{f_\pi}(\partial_\mu\pi^a)\sigma^a$$

$$- i\frac{1}{6}\frac{\pi^2}{f_\pi^3}(\partial_\mu\pi^a)\sigma^a + i\frac{1}{6}\frac{\pi}{f_\pi^3}\pi^a\sigma^a\partial_\mu\pi + \mathcal{O}\left(\frac{1}{f_\pi^4}\right)$$

- $$\mathcal{L} = \frac{f_\pi^2}{4}\,\mathrm{Tr}\left((\partial_\mu U)^\dagger(\partial^\mu U)\right)$$

$$= \frac{f_\pi^2}{4}\,\mathrm{Tr}\Big(\frac{\pi^2}{f_\pi^4}(\partial_\mu\pi)(\partial^\mu\pi) - 2\cdot\frac{1}{2}\frac{\pi}{f_\pi^4}\pi^a\sigma^a(\partial_\mu\pi)(\partial^\mu\pi^b)\sigma^b$$

$$+ \frac{1}{f_\pi^2}(\partial_\mu\pi^a)(\partial^\mu\pi^b)\sigma^a\sigma^b - \frac{1}{3}\frac{\pi^2}{f_\pi^4}(\partial_\mu\pi^a)(\partial^\mu\pi^b)\sigma^a\sigma^b$$

$$+ \frac{1}{3}\frac{\pi}{f_\pi^4}\pi^a\sigma^a\sigma^b(\partial_\mu\pi^b)(\partial_\mu\pi)\Big) + \mathcal{O}\left(\frac{1}{f_\pi^4}\right)$$

$$= \frac{1}{4}\cdot\overset{\mathrm{Tr}(\mathbb{1})}{2}\,\frac{1}{f_\pi^2}\cancel{\pi^2(\partial_\mu\pi)}(\partial^\mu\pi) - \frac{1}{2}\frac{1}{f_\pi^2}\pi\,\cancel{\pi^a(\partial_\mu\pi^a)}\underbrace{(\partial^\mu\pi)}_{=\frac{1}{\pi}\pi^a\partial^\mu\pi^a}$$

$$+ \frac{1}{2}(\partial_\mu\pi^a)(\partial^\mu\pi^a) - \frac{1}{6}\frac{1}{f_\pi^2}\pi^2(\partial_\mu\pi^a)(\partial^\mu\pi^a)$$

$$+ \frac{1}{6}\frac{1}{f_\pi^2}\pi(\pi^a\partial_\mu\pi^a)(\partial^\mu\pi) + \mathcal{O}\left(\frac{1}{f_\pi^4}\right)$$

$$= \frac{1}{2}(\partial_\mu\pi^a)(\partial^\mu\pi^a) + \frac{1}{6f_\pi^2}\left[(\pi^a\partial_\mu\pi^a)^2 - \pi^2(\partial_\mu\pi^a)(\partial^\mu\pi^a)\right]$$

$$+ \mathcal{O}\left(\frac{1}{f_\pi^4}\right)$$

(3)

- explicit symmetry breaking:
  choose vacuum $\langle U \rangle = \mathbb{1}$

- $\langle U \rangle \longmapsto V \mathbb{1} V^\dagger = \mathbb{1} = \langle U \rangle$
  Note that $SU(2)_L \times SU(2)_R \times U(1)$ is not a symmetry anymore.

- $\pi_3 \left(\frac{SU(2)_L \times SU(2)_R \times U(1)}{SU(2)_V} \right) \simeq \pi_3 \left(SU(2) \times U(1) \right) \simeq \pi_3 \left(S^3 \right) = \mathbb{Z}$

  $\rightarrow$ Third homotopy group is non-trivial

(4)

- $\pi^a(x) = f_\pi \, F(r) \, \frac{x^a}{r}$ (independent of time)

- finite energy means that the field goes to the vacuum state at infinity, i.e.
$$U(r \to \infty) = \langle U \rangle = \mathbb{1}$$

- $U(x) = \cos(F(r)) \, \mathbb{1} + i \sin(F(r)) \frac{x^a}{r} \sigma^a$

  $\rightarrow F(r \to \infty) = n \cdot \pi$ with $n \in \mathbb{Z}$

- To have a continuous function we additionally have to require $F(r \to 0) = m \cdot \pi$ with $m \in \mathbb{Z}$

(5)

- $U(x) = \cos(F(r)) \, \mathbb{1} + i \sin(F(r)) \, \frac{x^a}{r} \sigma^a$
  $\partial_i U(x) = -\sin(F) \frac{x_i}{r} F' + i \cos(F) \frac{x^a x^i}{r^2} F' \sigma^a + i \sin(F) \frac{1}{r} \left(\delta_{ai} - \frac{x^a x^i}{r^2} \right) \sigma^a$

$$\partial_i U(x) = -\sin(F)\,\frac{x_i}{r}\,F' + i\cos(F)\,\frac{x^a x^i}{r^2}\,F'\sigma^a + i\sin(F)\frac{1}{r}\left(\delta_{ai} - \frac{x^a x^i}{r^2}\right)\sigma^a$$

$$\begin{aligned}
U^\dagger \partial_i U = & -\cos(F)\cancel{\sin(F)}\,\frac{x_i}{r}\,F' + i\cos^2(F)\,\frac{x^a x^i}{r^2}\,F'\sigma^a \\
& + i\cos(F)\sin(F)\,\frac{1}{r}\left(\delta_{ai} - \frac{x^a x^i}{r^2}\right)\sigma^a + i\sin^2(F)\,\frac{x^a x^i}{r^2}\,F'\sigma^a \\
& + \cos(F)\cancel{\sin(F)}\,\frac{x^i}{r}\,F' + \sin^2(F)\left(\delta_{ai} - \frac{x^a x^i}{r^2}\right)\frac{x^b}{r^2}\,\sigma^b\sigma^a
\end{aligned}$$

$\sigma^a\sigma^b = \delta_{ab} + i\varepsilon_{abc}\sigma^c$

$$\begin{aligned}
& = i\,\frac{x^a x^i}{r^2}\,F'\sigma^a + i\cos(F)\sin(F)\,\frac{1}{r}\left(\delta_{ai} - \frac{x^a x^i}{r^2}\right)\sigma^a \\
& + i\sin^2(F)\,\frac{x^b}{r^2}\,\varepsilon_{iba}\,\sigma^a \\
& = M_{ia}\,\sigma^a
\end{aligned}$$

- $B[U] = \frac{1}{24\pi^2}\int d^3x\ \varepsilon_{ijk}\,\mathrm{Tr}\left(U^\dagger(\partial_i U)\,U^\dagger(\partial_j U)\,U^\dagger(\partial_k U)\right)$

$\mathrm{Tr}(\sigma^a\sigma^b\sigma^c) = 2i\varepsilon_{abc}$

$$= \frac{2i}{24\pi^2}\int d^3x\ \underbrace{\varepsilon_{ijk}\,\varepsilon_{abc}\,M_{ia}\,M_{jb}\,M_{kc}}$$

all indices contracted and
M_{ia} depends only on x^i, x^a
→ The integrand is spherically symmetric

- We can choose $\frac{x_i}{r} = (0,0,1)$

$$M_{ia} = i\begin{pmatrix} \frac{\cos(F)\sin(F)}{r} & -\frac{1}{r}\sin^2(F) & 0 \\ \frac{1}{r}\sin^2(F) & \frac{\cos(F)\sin(F)}{r} & 0 \\ 0 & 0 & F' \end{pmatrix}$$

- $\det M = -i\left(\frac{1}{r^2}\cos^2(F)\sin^2(F)\,F' + \frac{1}{r^2}\sin^4(F)\,F'\right)$
$\qquad = -i\,\frac{1}{r^2}\,F'\sin^2(F)$

- Remember $\det A = \frac{1}{3!}\,\varepsilon_{abc}\,\varepsilon_{ijk}\,A_{ai}\,A_{bj}\,A_{ck}$ (in 3d)

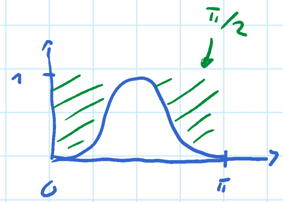
→ $B[U] = \frac{i}{2\pi^2}\int d^3x\ \det M$

$$\to B[U] = \frac{i}{2\pi^2}\int d^3x \ \det M$$

$$= \frac{1}{2\pi^2}\, 4\pi \int_0^\infty dr \ F' \sin^2(F) \qquad \text{with} \quad dF = F' dr$$

$$= \frac{2}{\pi}\int_{m\cdot\pi}^{n\cdot\pi} dF \ \sin^2(F)$$

$$= \frac{2}{\pi}\int_0^{(n-m)\cdot\pi} dF \ \sin^2(F)$$



$$= \frac{2}{\pi}\,(n-m)\cdot\frac{\pi}{2} = n-m \in \mathbb{Z}$$

Problem 2:

(1)

- Another term could be for example

$$[\mathrm{Tr}(\partial_\mu U \, \partial^\mu U^+)]^2$$

- $\mathrm{Tr}(\sigma^a\sigma^b\sigma^c\sigma^d)$

$$= 2\delta_{ab}\delta_{cd} - 2\delta_{ac}\delta_{bd} + 2\delta_{ad}\delta_{bc}$$

$$= \frac{1}{2}\mathrm{Tr}(\sigma^a\sigma^b)\,\mathrm{Tr}(\sigma^c\sigma^d) - \frac{1}{2}\mathrm{Tr}(\sigma^a\sigma^c)\,\mathrm{Tr}(\sigma^b\sigma^d) + \frac{1}{2}\mathrm{Tr}(\sigma^a\sigma^d)\,\mathrm{Tr}(\sigma^b\sigma^c)$$

- $[\mathrm{Tr}(\partial_\mu U \, \partial^\mu U^+)]^2$

$$\sim (\partial_\mu\pi^a)(\partial^\mu\pi^b)(\partial_\nu\pi^c)(\partial^\nu\pi^d)\big(\mathrm{Tr}(\sigma^a\sigma^b)\,\mathrm{Tr}(\sigma^c\sigma^d) - \mathrm{Tr}(\sigma^a\sigma^c)\,\mathrm{Tr}(\sigma^b\sigma^d) + \mathrm{Tr}(\sigma^a\sigma^d)\,\mathrm{Tr}(\sigma^b\sigma^c)\big)$$

$$\sim \mathrm{Tr}\big(\sigma^a(\partial_\mu\pi^a)\,\sigma^b(\partial^\mu\pi^b)\,\sigma^c(\partial_\nu\pi^c)\,\sigma^d(\partial^\nu\pi^d)\big)$$

$$\sim \mathrm{Tr}(\partial_\mu U\,\partial^\mu U^+\,\partial_\nu U\,\partial^\nu U^+)$$

$$= (a)$$

(2)

- $[U^+\partial_\mu U, U^+\partial_\nu U] = U^+(\partial_\mu U)U^+(\partial_\nu U) - U^+(\partial_\nu U)U^+(\partial_\mu U)$

(2)

- $[U^\dagger \partial_\mu U, U^\dagger \partial_\nu U] = U^\dagger(\partial_\mu U)U^\dagger(\partial_\nu U) - U^\dagger(\partial_\nu U)U^\dagger(\partial_\mu U)$

$$= -(\partial_\mu U^\dagger)UU^\dagger(\partial_\nu U) + (\partial_\nu U^\dagger)UU^\dagger(\partial_\mu U)$$

$$= -(\partial_\mu U^\dagger)(\partial_\nu U) + (\partial_\nu U^\dagger)(\partial_\mu U)$$

- $\mathrm{Tr}\left([U^\dagger \partial_\mu U, U^\dagger \partial_\nu U]^2\right)$

$$= \mathrm{Tr}\Big((\partial_\mu U^\dagger)(\partial_\nu U)(\partial^\mu U^\dagger)(\partial^\nu U)$$

$$- (\partial_\mu U^\dagger)(\partial_\nu U)(\partial^\nu U^\dagger)(\partial^\mu U)$$

$$- (\partial_\nu U^\dagger)(\partial_\mu U)(\partial^\mu U^\dagger)(\partial^\nu U)$$

$$+ (\partial_\mu U^\dagger)(\partial_\nu U)(\partial^\mu U^\dagger)(\partial^\nu U)\Big)$$

$$= 2\,\mathrm{Tr}\Big((\partial_\mu U^\dagger)(\partial_\nu U)(\partial^\mu U^\dagger)(\partial^\nu U)$$

$$- (\partial_\mu U)(\partial^\mu U^\dagger)(\partial_\nu U)(\partial^\nu U^\dagger)\Big)$$

$$= 2\big((b) - (a)\big)$$

$\rightarrow$ $(a) - (b) = -\frac{1}{2}\mathrm{Tr}\left([U^\dagger \partial_\mu U, U^\dagger \partial_\nu U]^2\right)$

(3)

- $E[U] = \int d^3x \left(\frac{f_\pi^2}{4} \mathrm{Tr}\left((\partial_i U)(\partial_i U^\dagger)\right) - \frac{1}{32g^2} \mathrm{Tr}\left([U^\dagger \partial_i U, U^\dagger \partial_j U]^2\right)\right)$

$$= \int d^3x\, \mathrm{Tr}\Big(\left(\frac{f_\pi}{2}\partial_i U\right)^\dagger\left(\frac{f_\pi}{2}\partial_i U\right)$$

$$- \frac{1}{16g^2}(\partial_i U)(\partial_j U^\dagger)(\partial_i U)(\partial_j U^\dagger)$$

$$+ \frac{1}{16g^2}(\partial_i U)(\partial_i U^\dagger)(\partial_j U)(\partial_j U^\dagger)\Big)$$

$$= \int d^3x\, \mathrm{Tr}\Big(\left(\frac{f_\pi}{2}\partial_i U\right)^\dagger\left(\frac{f_\pi}{2}\partial_i U\right)$$

$$- \left(\frac{1}{4g} \varepsilon_{kij} (\partial_i U^\dagger)(\partial_j U) \right) \left(\frac{1}{4g} \varepsilon_{kmn} (\partial_m U^\dagger)(\partial_n U) \right))$$

$$= \int d^3x \; \mathrm{Tr} \Big(\left(\tfrac{f_\pi}{2} \partial_i U \right)^\dagger \left(\tfrac{f_\pi}{2} \partial_i U \right)$$

$$+ \left(\frac{1}{4g} \varepsilon_{kij} \; U (\partial_i U^\dagger)(\partial_j U) \right)^\dagger \left(\frac{1}{4g} \varepsilon_{kmn} \; U (\partial_m U^\dagger)(\partial_n U) \right) \Big)$$

$$= \int d^3x \; \mathrm{Tr} \Big[\Big(\frac{f_\pi}{2} \partial_k U \pm \frac{1}{4g} \varepsilon_{kij} U (\partial_i U^\dagger)(\partial_j U) \Big)^\dagger$$

$$\cdot \underbrace{\Big(\frac{f_\pi}{2} \partial_k U \pm \frac{1}{4g} \varepsilon_{kmn} U (\partial_m U^\dagger)(\partial_n U) \Big)}_{=: M}$$

$$\mp \frac{f_\pi}{8g} \varepsilon_{kij} \; (\partial_k U^\dagger) U (\partial_i \overset{UU^\dagger}{\downarrow} U^\dagger)(\partial_j U)$$

$$\mp \frac{f_\pi}{8g} \varepsilon_{kij} \; (\partial_j U^\dagger)(\partial_i U) \; U^\dagger (\partial_k U) \Big]$$

$$= \int d^3x \; \Big(\mathrm{Tr}(M^\dagger M) \mp \frac{f_\pi}{4g} \varepsilon_{kij} \; U^\dagger(\partial_k U) U^\dagger (\partial_i U) U^\dagger (\partial_j U) \Big)$$

$$= \underbrace{\Big(\int d^3x \; \mathrm{Tr}(M^\dagger M) \Big)}_{\geq 0} \mp 6\pi^2 \frac{f_\pi}{g} \; B[U]$$

$$\rightarrow E[U] \geq 6\pi^2 \frac{f_\pi}{g} \; |B[U]|$$

(4)

$$\bullet \; \mathcal{L}_{mass} = \frac{m_\pi^2 f_\pi^2}{4} \; \mathrm{Tr} \left(U + U^\dagger - 2 \right)$$

$$= \frac{m_\pi^2 f_\pi^2}{4} \; \mathrm{Tr} \Big(\cos\left(\frac{\pi}{f_\pi}\right) \cancel{+ i \sin\left(\frac{\pi}{f_\pi}\right)} \frac{\pi^a}{\pi} \sigma^a$$

$$+ \cos\left(\frac{\pi}{f_\pi}\right) \cancel{- i \sin\left(\frac{\pi}{f_\pi}\right)} \frac{\pi^a}{\pi} \sigma^a - 2 \Big)$$

$$= \frac{m_\pi^2 f_\pi^2}{4} \cdot 4 \left(\cos\left(\frac{\pi}{f_\pi}\right) - 1\right)$$

$$= m_\pi^2 f_\pi^2 \left(1 - \frac{1}{2} \frac{\pi^2}{f_\pi^2} - 1\right)$$

$$= -\frac{1}{2} m_\pi^2 \pi^a \pi^a$$

- $\mathrm{Tr}(U + U^\dagger - 2)$ has the global symmetry $SU(2)_V$: $U \longmapsto V U V^\dagger$

  → A mass term breaks the $SU(2)_L \times SU(2)_R$ symmetry

Comments

- Skyrmions play the role of pions in this model
- If we increase the flavor number i.e. the symmetry of the theory. For example we take U to be an $SU(3)$ field, then the model can describe baryons and $B[U]$ is the baryon number.

  In other words, baryons are skyrmions of of a sigma model effective field theory.