Topological Defects

Problem Sheet 13

29 January 2024

1. Linear Sigma Model

Let us analyze the $3+1$ dimensional Lagrangian

$$\mathcal{L} = \frac{f_\pi^2}{4}\,\mathrm{Tr}\big((\partial_\mu U^\dagger)(\partial^\mu U)\big), \tag{1}$$

with

$$U(x) = \exp\left(\frac{i}{f_\pi}\pi^a(x)\sigma^a\right) \tag{2}$$

being an $SU(2)$ field.

1. What is the global symmetry of this theory?
2. Show that you can rewrite the Lagrangian to

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \pi^a)(\partial^\mu \pi^a) + \frac{1}{6f_\pi^2}\left[(\pi^a\partial_\mu\pi^a)^2 - \pi^2(\partial_\mu\pi^a)(\partial^\mu\pi^a)\right] + \mathcal{O}\left(\frac{1}{f_\pi^4}\right), \tag{3}$$

   where $\pi = \sqrt{\pi^a\pi^a}$.

We can break the symmetry explicitly by choosing the vacuum to be $\langle U\rangle = \mathbb{1}$. The symmetry of the Lagrangian in the vacuum state is then $SU(2)_V : U \mapsto VUV^\dagger$.

3. Can you find a non-trivial homotopy group of the vacuum manifold?
4. Let us write $\pi^a(x) = f_\pi F(r)\frac{x^a}{r}$. What are the boundary conditions ($r \to 0$ and $r \to \infty$) for $F(r)$?

Let us define the charge

$$B[U] = \frac{1}{24\pi^2}\varepsilon_{ijk}\int \mathrm{d}^3x\,\mathrm{Tr}\big(U^\dagger(\partial_i U)U^\dagger(\partial_j U)U^\dagger(\partial_k U)\big). \tag{4}$$

5. Show that this charge is integer-valued by inserting the result of the previous part.

2. The Skyrme Model

By Derick's theorem, solitons in a renormalizable non-gauged theory in $3+1$ dimensions are not allowed. However, we can treat our theory as an effective field theory and add higher-order derivative terms

$$(a)\,\mathrm{Tr}\big((\partial_\mu U)(\partial^\mu U^\dagger)(\partial_\nu U)(\partial^\nu U^\dagger)\big),$$
$$(b)\,\mathrm{Tr}\big((\partial_\mu U)(\partial_\nu U^\dagger)(\partial^\mu U)(\partial^\nu U^\dagger)\big).$$

These are the only linearly independent terms with the global symmetry that you found in problem 1.

1. Convince yourself that there are no more terms with four derivatives.
2. Show that

$$(a)-(b) = -\frac{1}{2}\,\mathrm{Tr}\Big(\big[U^\dagger\partial_\mu U, U^\dagger\partial_\nu U\big]^2\Big). \tag{5}$$

It turns out that a term of this form is the only choice that gives finite energy solutions. These solutions are called Skyrmions. The Skyrme Lagrangian is given by

$$\mathcal{L}_{\text{Skyrme}} = \frac{f_\pi^2}{4}\,\mathrm{Tr}\big((\partial_\mu U)(\partial^\mu U^\dagger)\big) + \frac{1}{32g^2}\,\mathrm{Tr}\Big(\big[U^\dagger\partial_\mu U, U^\dagger\partial_\nu U\big]^2\Big). \tag{6}$$

3. By completing the square show that the Bogomolny bound is

$$E[U] \geq 6\pi^2\frac{f_\pi}{g}|B[U]|, \tag{7}$$

where $B[U]$ is given in problem 1.

As we saw in problem 1, the Lagrangian can be expanded to

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\pi^a)(\partial^\mu\pi^a) + \mathcal{O}(\pi^4). \tag{8}$$

π^a can get mass by a term of the form

$$\mathcal{L}_{\text{mass}} = -\frac{1}{2}m_\pi^2\pi^a\pi^a. \tag{9}$$

4. Show that this term can come from

$$\mathcal{L}_{\text{mass}} = \frac{m_\pi^2 f_\pi^2}{4}\,\mathrm{Tr}\big(U+U^\dagger-2\big). \tag{10}$$

What is the global symmetry of this mass term ?