Problem 1:

$$S = \frac{1}{2}\int d^4x \, \mathrm{Tr}(G_{\mu\nu} G^{\mu\nu}) \qquad (+ \text{ sign} \to \text{Euclidean metric})$$

(1)

- action vanishes for pure gauge as shown on sheet 7.

$$\to \quad W_\mu \xrightarrow{r\to\infty} \frac{i}{g} U \partial_\mu U^\dagger$$

(2)

- $U^{(0)} = 1 \quad \to \quad \partial_\mu U^{(0)} = 0 \quad \to \quad n = 0$

- $U^{(1)} = \dfrac{x_4 + i x_i \sigma_i}{r} = \dfrac{x^\mu \tau_\mu}{r} \qquad$ with $\quad \tau_\mu = (\mathbb{1}, i\sigma_1, i\sigma_2, i\sigma_3)$

$$\partial_\mu U^{(1)} = \frac{\tau_\mu}{r} - \frac{x^\nu \tau_\nu x_\mu}{r^3} = \frac{\tau_\mu}{r} - U^{(1)} \frac{x_\mu}{r^2}$$

$$(\partial_\mu U^{(1)}) U^{(1)\dagger} = \frac{1}{r^2}(x^\alpha \tau_\alpha^\dagger \tau_\mu - x_\mu)$$

where we used
$$U^{(1)} U^{(1)\dagger} = \frac{1}{r^2}(x_4 + i x_i \sigma_i)(x_4 - i x_j \sigma_j)$$
$$= \frac{1}{r^2}\left((x_4)^2 + x_i x_j \frac{1}{2}\{\sigma_i, \sigma_j\}\right)$$
$$= 1$$

- $$n = \frac{1}{24\pi^2}\int_{S^3_\infty} \underbrace{d^3 S_\mu}_{= d\Omega \, x^\mu r^2}$$
$$\cdot \, \mathrm{Tr}\left[\varepsilon^{\mu\nu\alpha\beta}(\partial_\nu U^{(1)}) U^{(1)\dagger} (\partial_\alpha U^{(1)}) U^{(1)\dagger} (\partial_\beta U^{(1)}) U^{(1)\dagger}\right]$$

$$= \frac{1}{24\pi^2}\int d\Omega \, \frac{1}{r^4} \mathrm{Tr}\left[x_\mu \varepsilon^{\mu\nu\alpha\beta}(x^\gamma \tau_\gamma^\dagger \tau_\nu - x_\nu)(x^\delta \tau_\delta^\dagger \tau_\alpha - x_\alpha)(x^\lambda \tau_\lambda^\dagger \tau_\beta - x_\beta)\right]$$

$$= \frac{1}{24\pi^2}\int d\Omega\, \frac{1}{r^4}\, \mathrm{Tr}\left[x_\mu\, \varepsilon^{\mu\nu\alpha\beta}\; \tau_\gamma^+ \tau_\nu\, \tau_\delta^+ \tau_\alpha\, \tau_\lambda^+ \tau_\beta\; x^\gamma x^\delta x^\lambda\right]$$

$$\bullet\; A = \mathrm{Tr}\left[x_\mu\, \varepsilon^{\mu\nu\alpha\beta}\; \tau_\gamma^+ \tau_\nu\, \tau_\delta^+ \tau_\alpha\, \tau_\lambda^+ \tau_\beta\; x^\gamma x^\delta x^\lambda\right]$$

$$= \mathrm{Tr}\left[x_4\, \varepsilon_{ijk}\; \tau_\gamma^+ \tau_i\, \tau_\delta^+ \tau_j\, \tau_\lambda^+ \tau_k\; x^\gamma x^\delta x^\lambda\right]$$

$$+ 3\, \mathrm{Tr}\left[x_i\, \varepsilon_{i4jk}\; \tau_\gamma^+ \tau_\delta^+ \tau_j\, \tau_\lambda^+ \tau_k\; x^\gamma x^\delta x^\lambda\right]$$

$$= \mathrm{Tr}\left[x_4\, \varepsilon_{ijk}\; \frac{1}{2}\left[x^\gamma \tau_\gamma^+ \tau_i\,,\, x^\delta \tau_\delta^+ \tau_j\right] \tau_\lambda^+ \tau_k\; x^\lambda\right]$$

$$- 3\, \mathrm{Tr}\left[x_i\, \varepsilon_{ijk}\; \frac{1}{2}\, \tau_\gamma^+ \left[x^\delta \tau_\delta^+ \tau_j\,,\, x^\lambda \tau_\lambda^+ \tau_k\right] x^\gamma\right]$$

$$\bullet\; \left[x^\gamma \tau_\gamma^+ \tau_j\,,\, x^\delta \tau_\delta^+ \tau_k\right]$$

$$= \left[x^4\, i\sigma_j + x_m \sigma_m \sigma_j\,,\, x^4\, i\sigma_k + x_n \sigma_n \sigma_k\right]$$

$\sigma_a \sigma_b = \delta_{ab} + i\varepsilon_{abc}\sigma_c$

$$\overset{?}{=} \left[x^4\, i\sigma_j + \cancel{x_j} + i\, x_m \varepsilon_{mja}\sigma_a\,,\, x^4\, i\sigma_k + \cancel{x_k} + i\, x_n \varepsilon_{nkb}\sigma_b\right]$$

$$= -\left(x^4 \delta_{aj} + x_m \varepsilon_{mja}\right)\left(x^4 \delta_{bk} + x_n \varepsilon_{nkb}\right) \underbrace{\left[\sigma_a, \sigma_b\right]}_{= i\, 2\, \varepsilon_{abc}\sigma_c}$$

$$= -2i\, \left(x^4 \varepsilon_{jbc} + x_m\left(\delta_{mb}\delta_{jc} - \delta_{mc}\delta_{bj}\right)\right)\left(x^4 \delta_{bk} + x_n \varepsilon_{nkb}\right) \sigma_c$$

$$= -2i\, \left(x^4 \varepsilon_{jbc} + x_b \delta_{jc} - x_c \delta_{jb}\right)\left(x^4 \delta_{bk} + x_n \varepsilon_{nkb}\right) \sigma_c$$

$$= -2i\, \Big(x^4 x^4 \varepsilon_{jkc} + x^4 x_k \delta_{jc} - x^4 \cancel{x_c}\, \delta_{jk}$$

$$+ \underbrace{x^4 x_n\, \varepsilon_{jbc}\varepsilon_{nkb}}_{= + x^4\, \cancel{x_c \delta_{jk}} - x^4 x_j \delta_{kc}} - x_n x_c\, \varepsilon_{nkj}\Big)\sigma_c$$

$$\varepsilon_{ijk}\left[x^\gamma \tau_\gamma^+ \tau_j\,,\, x^\delta \tau_\delta^+ \tau_k\right]$$

$$= -4i\, \left(x^4 x^4 \delta_{ic} - x^4 x_k\, \varepsilon_{ikc} + x_i x_c\right)\sigma_c$$

$$x_i\, \varepsilon_{ijk}\left[x^\gamma \tau_\gamma \tau_j\,,\, x^\delta \tau_\delta \tau_k\right]$$

$$= -4i\, r^2\, x_c \sigma_c$$

$$\bullet\ A = \mathrm{Tr}\left[x_4\, \varepsilon_{ijk}\, \frac{1}{2} \left[x^\gamma \tau_\gamma^\dagger \tau_i ,\, x^\delta \tau_\delta^\dagger \tau_j\right] \tau_\lambda^\dagger \tau_k\, x^\lambda\right]$$
$$- 3\, \mathrm{Tr}\left[x_i\, \varepsilon_{ijk}\, \frac{1}{2}\, \tau_\gamma^\dagger \left[x^\delta \tau_\delta^\dagger \tau_j ,\, x^\lambda \tau_\lambda^\dagger \tau_k\right] x^\gamma\right]$$

$$= 2\, x_4 \left(x^4 x^4 \delta_{kc} - x^4 x_m \varepsilon_{kmc} + x_k x_c\right) \mathrm{Tr}\left[\sigma_c\, \tau_\lambda^\dagger\, \sigma_k\, x^\lambda\right]$$
$$+ 3 \cdot 2i\, r^2\, x_c\, \mathrm{Tr}\left[\tau_\gamma^\dagger\, \sigma_c\, x^\gamma\right]$$

With $\mathrm{Tr}(\sigma_j \sigma_k) = 2\delta_{jk}$, $\mathrm{Tr}(\sigma_i \sigma_j \sigma_k) = 2i\, \varepsilon_{ijk}$

$$A = 4\, x_4 \left(x^4 x^4 \delta_{kc} - x^4 x_m \varepsilon_{kmc} + x_k x_c\right) \left(\delta_{ck}\, x^4 + \varepsilon_{cnk}\, x^n\right)$$
$$+ 12\, r^2\, x_c\, \delta_{cm}\, x^m$$
$$= 4\, x_4 \left(3\, (x_4)^3 + x_k x^k x_4 + 2\, x^4 x_k x^k\right)$$
$$+ 12\, r^2\, x_k x^k$$
$$= 12\, (x_4)^4 + 12\, (x_4)^2 (x_k x^k) + 12\, r^2 (x_k x^k)$$
$$= 12\, r^4$$

$$\rightarrow n = \frac{1}{24\pi^2} \underbrace{\int d\Omega}_{= 2\pi^2} \frac{1}{r^4}\, 12\, r^4 = 1$$

• Now we can show by induction that $U^{(k)}$ will give a winding k.

$$U_{k-1}^\dagger\, \partial_\mu U_{k-1} = U_{k-1}^\dagger \left[(\partial_\mu U_1) U_{k-2} + U_1 (\partial_\mu U_1) U_{k-3} + \dots + U_{k-2}\, \partial_\mu U_1\right]$$

$$U_k^\dagger\, \partial_\mu U_k = U_k^\dagger \left[(\partial_\mu U_1) U_{k-1} + U_1 (\partial_\mu U_1) U_{k-2} + \dots + U_{k-1}\, \partial_\mu U_1\right]$$
$$= U_1^\dagger \Big(U_{k-1}^\dagger \big[(\partial_\mu U_1) U_{k-2} + U_1 (\partial_\mu U_1) U_{k-3}$$

$$= U_1^+ \Big(U_{k-1}^+ \big[(\partial_\mu U_1) U_{k-2} + U_1 (\partial_\mu U_1) U_{k-3} + \dots + U_{k-2} \partial_\mu U_1 \big] \Big) U_1 + U_1^+ \partial_\mu U_1$$

$$= U_1^+ \left(U_{k-1}^+ \partial_\mu U_{k-1} \right) U_1 + U_1^+ \partial_\mu U_1$$

- $$n_k = \frac{1}{24\pi^2} \int d^3 S_\mu \, \varepsilon^{\mu\nu\alpha\beta} \, \mathrm{Tr}\left[(U_k^+ \partial_\nu U_k)(U_k^+ \partial_\alpha U_k)(U_k^+ \partial_\beta U_k) \right]$$

$$= n_{k-1} + 1 + \frac{1}{24\pi^2} \int d^3 S_\mu \, \varepsilon^{\mu\nu\alpha\beta} \cdot$$

$$\cdot \mathrm{Tr}\Big[3 U_1^+ (U_{k-1}^+ \partial_\nu U_{k-1}) U_1 U_1^+ (\partial_\alpha U_1) \underbrace{U_1^+ \partial_\beta U_1}_{= -(\partial_\beta U_1^+) U_1}$$
$$+ 3 U_1^+ (U_{k-1}^+ \partial_\nu U_{k-1}) U_1 U_1^+ (U_{k-1}^+ \partial_\alpha U_{k-1}) U_1 U_1^+ \partial_\beta U_1 \Big]$$

$$= n_{k-1} + 1 + \frac{3}{24\pi^2} \int d^3 S_\mu \, \varepsilon^{\mu\nu\alpha\beta} \cdot$$

$$\cdot \mathrm{Tr}\Big[- U_{k-1}^+ (\partial_\nu U_{k-1}) (\partial_\alpha U_1) (\partial_\beta U_1^+)$$
$$- U_1^+ (\partial_\nu U_{k-1}^+) (\partial_\alpha U_{k-1}) (\partial_\beta U_1) \Big]$$

product rule

$$= n_{k-1} + 1 - \frac{1}{4\pi^2} \underbrace{\int d^3 S_\mu \, \varepsilon^{\mu\nu\alpha\beta} \, \mathrm{Tr}\left[(\partial_\nu U_{k-1})(\partial_\alpha U_1)(\partial_\beta U_k^+) \right]}_{= \int d^4 x \, \varepsilon^{\mu\nu\alpha\beta} \partial_\mu \mathrm{Tr}\left[(\partial_\nu U_{k-1})(\partial_\alpha U_1)(\partial_\beta U_k^+) \right] = 0}$$

$$= n_{k-1} + 1$$

$\rightarrow$ $U^{(k)}$ gives winding k

(3)

- $$\mathrm{Tr}\left[G_{\mu\nu} \tilde{G}^{\mu\nu} \right] = \frac{1}{2} \varepsilon^{\mu\nu\alpha\beta} \, \mathrm{Tr}\Big[(\partial_\mu W_\nu - \partial_\nu W_\mu - ig [W_\mu, W_\nu]) \cdot (\partial_\alpha W_\beta - \partial_\beta W_\alpha - ig [W_\alpha, W_\beta]) \Big]$$

$$= \varepsilon^{\mu\nu\alpha\beta}\,\mathrm{Tr}\left[(\partial_\mu W_\nu - ig W_\mu W_\nu)(\partial_\alpha W_\beta - ig W_\alpha W_\beta)\right]$$

$$= \varepsilon^{\mu\nu\alpha\beta}\,\mathrm{Tr}\Big[(\partial_\mu W_\nu)(\partial_\alpha W_\beta) - g^2 W_\mu W_\nu W_\alpha W_\beta$$
$$- ig\, W_\mu W_\nu\, \partial_\alpha W_\beta - ig\, W_\alpha W_\beta\, \partial_\mu W_\nu\Big]$$

$$= \varepsilon^{\mu\nu\alpha\beta}\,\mathrm{Tr}\left[(\partial_\mu W_\nu)(\partial_\alpha W_\beta) - 2ig\, W_\mu W_\nu\, \partial_\alpha W_\beta\right]$$

where we used $\varepsilon^{\mu\nu\alpha\beta}\,\mathrm{Tr}(W_\mu W_\nu W_\alpha W_\beta)$

$$= \varepsilon^{\cancel{\mu\nu\alpha\beta}}_{\beta\mu\nu\alpha}\,\mathrm{Tr}(W_{\underset{\mu}{\cancel{\alpha}}} W_{\underset{\nu}{\cancel{\alpha}}} W_{\underset{\alpha}{\cancel{\beta}}} W_{\underset{\beta}{\cancel{\mu}}})$$

$$= -\varepsilon^{\mu\nu\alpha\beta}\,\mathrm{Tr}(W_\mu W_\nu W_\alpha W_\beta) = 0$$

- $\mathrm{Tr}\left[G_{\mu\nu}\tilde{G}^{\mu\nu}\right] = \varepsilon^{\mu\nu\alpha\beta}\,\mathrm{Tr}\Big[\partial_\mu (W_\nu \partial_\alpha W_\beta)$

$$-\frac{4}{3} ig \Big(\varepsilon^{\cancel{\mu}\cancel{\nu}\cancel{\alpha}\beta}_{\nu\alpha\mu}\, W_{\underset{\nu}{\cancel{\mu}}} W_{\underset{\alpha}{\cancel{\nu}}} (\partial_{\underset{\mu}{\cancel{\alpha}}} W_\beta) + \varepsilon^{\cancel{\mu}\nu\cancel{\alpha}\cancel{\beta}}_{\beta\,\mu\alpha}\, W_\nu (\partial_{\underset{\mu}{\cancel{\alpha}}} W_{\underset{\alpha}{\cancel{\beta}}}) W_{\underset{\beta}{\cancel{\mu}}}$$
$$+ \varepsilon^{\cancel{\mu\nu\alpha\beta}}_{\alpha\beta\mu\nu}\, (\partial_{\underset{\mu}{\cancel{\alpha}}} W_{\underset{\nu}{\cancel{\beta}}}) W_{\underset{\alpha}{\cancel{\mu}}} W_{\underset{\beta}{\cancel{\nu}}} \Big)\Big]$$

$$= \varepsilon^{\mu\nu\alpha\beta}\,\partial_\mu\,\mathrm{Tr}\left[W_\nu \partial_\alpha W_\beta - \frac{4i}{3} g\, W_\nu W_\alpha W_\beta\right]$$

- With $W_\mu = \frac{i}{g} U \partial_\mu U^\dagger$

$\mathrm{Tr}\left[G_{\mu\nu}\tilde{G}^{\mu\nu}\right]$

$$= \varepsilon^{\mu\nu\alpha\beta}\,\partial_\mu\,\mathrm{Tr}\Big[-\frac{1}{g^2} U(\partial_\nu U^\dagger)\,\partial_\alpha(U\partial_\beta U^\dagger)$$
$$- \frac{4}{3g^2} U(\partial_\nu U^\dagger)\, U(\partial_\alpha U^\dagger)\, U(\partial_\beta U^\dagger)\Big]$$

$$= -\frac{1}{g^2}\varepsilon^{\mu\nu\alpha\beta}\,\partial_\mu\,\mathrm{Tr}\Big[U(\partial_\nu U^\dagger)\,\overbrace{(\partial_\alpha U) U^\dagger}^{= -U\partial_\alpha U^\dagger}\, U(\partial_\beta U^\dagger)$$
$$+ \frac{4}{3} U(\partial_\nu U^\dagger)\, U(\partial_\alpha U^\dagger)\, U(\partial_\beta U^\dagger)\Big]$$

$$= -\frac{1}{3g^2}\,\varepsilon^{\mu\nu\alpha\beta}\,\partial_\mu\,\mathrm{Tr}\left[U(\partial_\nu U^\dagger)\, U(\partial_\alpha U^\dagger)\, U(\partial_\beta U^\dagger)\right]$$

$$= \frac{1}{3g^2} \varepsilon^{\mu\nu\alpha\beta} \partial_\mu \mathrm{Tr}\left[(\partial_\nu U)U^\dagger(\partial_\alpha U)U^\dagger(\partial_\beta U)U^\dagger\right]$$

- $$n = \frac{1}{24\pi^2}\int d^3S_\mu \ \mathrm{Tr}\left[\varepsilon^{\mu\nu\alpha\beta}(\partial_\nu U)U^\dagger(\partial_\alpha U)U^\dagger(\partial_\beta U)U^\dagger\right]$$
$$= \frac{1}{24\pi^2}\int d^4x \quad 3g^2\, \mathrm{Tr}\left[G_{\mu\nu}\tilde{G}^{\mu\nu}\right]$$
$$= \frac{g^2}{8\pi^2}\int d^4x \ \mathrm{Tr}\left[G_{\mu\nu}\tilde{G}^{\mu\nu}\right]$$

(4)

- $$\mathrm{Tr}\left[G_{\mu\nu}\tilde{G}^{\mu\nu}\right] = \varepsilon^{\mu\nu\alpha\beta}\partial_\mu \mathrm{Tr}\left[W_\nu \partial_\alpha W_\beta - \frac{4i}{3}g\, W_\nu W_\alpha W_\beta\right]$$
$$= \varepsilon^{\mu\nu\alpha\beta}\partial_\mu \mathrm{Tr}\Big[W_\nu G_{\alpha\beta} + ig\, W_\nu[W_\alpha, W_\beta] - \frac{4i}{3}g\, W_\nu W_\alpha W_\beta\Big]$$
$$= \varepsilon^{\mu\nu\alpha\beta}\partial_\mu \mathrm{Tr}\left[W_\nu G_{\alpha\beta} + \frac{2i}{3}g\, W_\nu W_\alpha W_\beta\right]$$

- $$S = \frac{1}{2}\int d^4x\ \mathrm{Tr}(G_{\mu\nu}G^{\mu\nu})$$
$$= \frac{1}{4}\int d^4x\ \underbrace{\mathrm{Tr}(G_{\mu\nu} \mp \tilde{G}_{\mu\nu})^2}_{\geq 0} \pm 2\,\mathrm{Tr}(G_{\mu\nu}\tilde{G}^{\mu\nu}) \qquad (1)$$
$$\geq \pm\frac{1}{2}\int d^4x\ \mathrm{Tr}(G_{\mu\nu}\tilde{G}^{\mu\nu})$$
$$= \pm\frac{1}{2}\int d^4x\ \varepsilon^{\mu\nu\alpha\beta}\partial_\mu \mathrm{Tr}\left[W_\nu G_{\alpha\beta} + \frac{2i}{3}g\, W_\nu W_\alpha W_\beta\right] = \pm\frac{4\pi^2}{g^2}n$$

$$\Gamma \quad \tilde{G}_{\mu\nu}\tilde{G}^{\mu\nu} = \frac{1}{4}\varepsilon_{\mu\nu\alpha\beta}\varepsilon^{\mu\nu\gamma\delta}G^{\alpha\beta}G_{\gamma\delta}$$
$$= \frac{1}{4}\cdot 2!\ \delta^{\gamma\delta}_{\alpha\beta}\, G^{\alpha\beta}G_{\gamma\delta}$$
$$= \frac{1}{2}\cdot(\delta^\gamma_\alpha\delta^\delta_\beta - \delta^\gamma_\beta\delta^\delta_\alpha)\, G^{\alpha\beta}G_{\gamma\delta}$$
$$= G_{\mu\nu}G^{\mu\nu}$$

(5)

- From (1) we see immediately that the inequality is an equality for

$$G_{\mu\nu} \pm \tilde{G}_{\mu\nu} = 0$$

- from

$$n = \frac{g^2}{16\pi^2} \int d^4x \; Tr(G_{\mu\nu} \tilde{G}^{\mu\nu})$$
$$= \mp \frac{g^2}{16\pi^2} \int d^4x \; \underbrace{Tr(\tilde{G}_{\mu\nu} \tilde{G}^{\mu\nu})}_{\geq 0}$$

we see that the upper/lower sign corresponds to a negative/positive winding number.

Problem 2:

(1)

- $W_\mu \xrightarrow{r\to\infty} \frac{i}{g} U \partial_\mu U^\dagger$ with $U = \frac{x_4 + i x_i \sigma_i}{r}$

- $$W_4 \xrightarrow{r\to\infty} \frac{i}{g} \cdot \frac{1}{r}(x_4 + i x_k \sigma_k) \cdot \left[\frac{1}{r} - \frac{x_4}{r^3}(x_4 - i x_j \sigma_j)\right]$$
$$= \frac{i}{g} \cdot \frac{1}{r^2}\left[(\cancel{x_4} + i x_k \sigma_k) - \frac{x_4}{r^2}\cancel{(x_4 x_4 + x_k x_k)}\right]$$
$$= -\frac{1}{g}\frac{1}{r^2} x_k \sigma_k = -\frac{2}{g}\frac{1}{r^2} x_a \underbrace{T_a}_{\text{generator of } SU(2)}$$
$$= -\frac{2}{g} \delta_{a\nu} \frac{x^\nu}{r^2} T_a$$
$$= \frac{2}{g} \eta_{a4\nu} \frac{x^\nu}{r^2} T_a$$

- $$W_k \xrightarrow{r\to\infty} \frac{i}{g} \cdot \frac{1}{r}(x_4 + i x_m \sigma_m) \cdot \left[\frac{-i\sigma_k}{r} - \frac{x_k}{r^3}(x_4 - i x_n \sigma_n)\right]$$
$$= \frac{i}{g} \cdot \frac{1}{r^2}\Big[-i x_4 \sigma_k + x_m \underbrace{\sigma_m \sigma_k}_{=\delta_{mk} + i\varepsilon_{mkn}\sigma_n} - x_4 x_4 \frac{x_k}{r^2} + \cancel{i x_4 \frac{1}{r^2} x_k x_n \sigma_n}$$
$$\cancel{- i x_4 \frac{1}{r^2} x_k x_m \sigma_m} - \frac{x_k}{r^2} x_m x_m\Big]$$

$= \frac{i}{g} \frac{1}{r} [$

$$\cdots \tfrac{\cdot}{4}\, \cancel{r^2}\, x_k\, x_m \sigma_m \quad \tfrac{\cdot}{r^2}\, x_m x_m \Big]$$

$$= \frac{i}{g} \cdot \frac{1}{r^2} \left[-i x_4 \sigma_k + x_k - i \varepsilon_{kmn} x_m \sigma_n - x_k \right]$$

$$= \frac{2}{g} \cdot \frac{1}{r^2} \left(x_4 \delta_{ka} + \varepsilon_{kma} x_m \right) T_a$$

$$= \frac{2}{g} \eta_{ak\nu} \frac{x^\nu}{r^2} T_a$$

$$\rightarrow W_\mu^a \xrightarrow{r \to \infty} \frac{2}{g} \eta_{a\mu\nu} \frac{x^\nu}{r^2}$$

(2)

- $$G_{\mu\nu}^a = \partial_\mu W_\nu^a - \partial_\nu W_\mu^a + g \varepsilon_{abc} W_\mu^b W_\nu^c$$

$$= \frac{2}{g} \eta_{a\nu\alpha} \left(\frac{\delta_{\mu\alpha}}{r^2} f - 2 \frac{x^\mu x^\alpha}{r^4} f + \frac{x^\mu x^\alpha}{r^3} f' \right)$$

$$- \frac{2}{g} \eta_{a\mu\alpha} \left(\frac{\delta_{\nu\alpha}}{r^2} f - 2 \frac{x^\nu x^\alpha}{r^4} f + \frac{x^\nu x^\alpha}{r^3} f' \right)$$

$$+ \frac{4}{g} \varepsilon_{abc} \eta_{b\mu\alpha} \eta_{c\nu\beta} \frac{x^\alpha x^\beta}{r^4} f^2$$

first identity

$$= -\underline{\frac{4}{g} \eta_{a\mu\nu} \frac{f}{r^2}} - \frac{2}{g} \frac{1}{r^4} \left(-2f + r f' \right) \left(\eta_{a\mu\alpha} x^\nu x^\alpha - \eta_{a\nu\alpha} x^\mu x^\alpha \right)$$

$$+ \frac{4}{g} \left(\cancel{\delta_{\mu\nu} \eta_{a\alpha\beta}} - \delta_{\mu\beta} \eta_{a\alpha\nu} - \delta_{\alpha\nu} \eta_{a\mu\beta} + \underline{\delta_{\alpha\beta} \eta_{a\mu\nu}} \right) \frac{x^\alpha x^\beta}{r^4} f^2$$

$$= -\frac{4}{g} \eta_{a\mu\nu} \frac{f(1-f)}{r^2} - \frac{2}{g} \frac{1}{r^4} \left(-2f + r f' \right) \left(\eta_{a\mu\alpha} x^\nu x^\alpha - \eta_{a\nu\alpha} x^\mu x^\alpha \right)$$

$$- \frac{4}{g} \frac{1}{r^4} f^2 \left(\eta_{a\mu\alpha} x^\nu x^\alpha + \eta_{a\alpha\nu} x^\mu x^\alpha \right)$$

$$= -\frac{4}{g} \left(\eta_{a\mu\nu} \frac{f(1-f)}{r^2} + \frac{1}{r^4} \left(-f + \frac{1}{2} r f' + f^2 \right) \left(\eta_{a\mu\alpha} x^\nu x^\alpha - \eta_{a\nu\alpha} x^\mu x^\alpha \right) \right)$$

- $$\tilde{G}_{\mu\nu}^a = \frac{1}{2} \varepsilon_{\mu\nu\alpha\beta} G_{\alpha\beta}^a$$

$$= -\frac{2}{g}\Big(\varepsilon_{\mu\nu\alpha\beta}\,\eta_{a\alpha\beta}\,\frac{f(1-f)}{r^2}$$

$$+ \frac{1}{r^4}\left(-f+\frac{1}{2}rf'+f^2\right)\varepsilon_{\mu\nu\alpha\beta}\left(\eta_{a\alpha\gamma}\,x^\beta x^\gamma - \eta_{a\beta\gamma}\,x^\alpha x^\gamma\right)\Big)$$

second identity

$$= -\frac{2}{g}\Big(\left(\delta_{\alpha\mu}\,\eta_{a\nu\alpha} - \delta_{\alpha\nu}\,\eta_{a\mu\alpha} + \delta_{\alpha\alpha}\,\eta_{a\mu\nu}\right)\frac{f(1-f)}{r^2}$$

$$+ \frac{1}{r^4}\left(-f+\frac{1}{2}rf'+f^2\right)\Big(-\left(\delta_{\gamma\mu}\,\eta_{a\nu\beta} - \delta_{\gamma\nu}\,\eta_{a\mu\beta} + \delta_{\gamma\beta}\,\eta_{a\mu\nu}\right)x^\beta x^\gamma$$

$$-\left(\delta_{\gamma\mu}\,\eta_{a\nu\alpha} - \delta_{\gamma\nu}\,\eta_{a\mu\alpha} + \delta_{\gamma\alpha}\,\eta_{a\mu\nu}\right)x^\alpha x^\gamma\Big)\Big)$$

$$= -\frac{2}{g}\Big(2\,\eta_{a\mu\nu}\,\frac{f(1-f)}{r^2}$$

$$+ \frac{1}{r^4}\left(-f+\frac{1}{2}rf'+f^2\right)\Big(-\left(\eta_{a\nu\alpha}\,x^\mu x^\alpha - \eta_{a\mu\alpha}\,x^\nu x^\alpha + r^2\,\eta_{a\mu\nu}\right)$$

$$-\left(\eta_{a\nu\alpha}\,x^\mu x^\alpha - \eta_{a\mu\alpha}\,x^\nu x^\alpha + r^2\,\eta_{a\mu\nu}\right)\Big)\Big)$$

$$= -\frac{4}{g}\left(\eta_{a\mu\nu}\,\frac{f(1-f)}{r^2} + \frac{1}{r^4}\left(-f+\frac{1}{2}rf'+f^2\right)\left(\eta_{a\mu\alpha}\,x^\nu x^\alpha - \eta_{a\nu\alpha}\,x^\mu x^\alpha\right)\right)$$

$$+ \frac{4}{g}\,\frac{1}{r^2}\left(-f+\frac{1}{2}rf'+f^2\right)\eta_{a\mu\nu}$$

$$= G^a_{\mu\nu} + \frac{4}{g}\,\frac{1}{r^2}\left(-f+\frac{1}{2}rf'+f^2\right)\eta_{a\mu\nu}$$

- self-duality equation gives

$$2f(f-1) + rf' = 0$$

(3)

- $$2f(1-f) = r\,\frac{df}{dr}$$

$$2\int \frac{1}{r}\,dr = \int \frac{1}{f(1-f)}\,df$$

$$2\cdot \ln(r) + \text{const} = \ln(f) - \ln(1-f)$$

$$2\cdot \ln(r) + \text{const} = \ln(f) - \ln(1-f)$$

$$\ln(r^2\cdot \text{const}) = \ln\left(\frac{f}{1-f}\right)$$

$$\frac{1}{r^2\cdot \text{const}} = \frac{1-f}{f}$$

$$\frac{1}{r^2\cdot \text{const}} = \frac{1}{f} - 1$$

$$f = \frac{1}{1+\frac{1}{r^2\cdot \text{const}}} = \frac{r^2}{r^2+\text{const}}$$

$\rightarrow W_\mu^a = \frac{2}{g}\eta_{a\mu\nu}\frac{x^\nu}{r^2+g^2}$ where g is an integration constant.

Comment: Note that W_μ^a falls of with $\frac{1}{r}$.
However, we can get rid of this $\frac{1}{r}$ behaviour by doing a gauge transformation.
Such a gauge is then called singular gauge. For more information see for example Shifman.

Problem 3:

transition amplitude: $\langle \Phi_F | e^{iHT} | \Phi_I \rangle = \int D\phi \, e^{iS}$

Wick rotation $T \rightarrow iT_E$:

$$\langle \Phi_F | e^{-HT_E} | \Phi_I \rangle = \int D\Phi \, e^{-S_E}$$

(1)

- We know that the action $S = -\frac{1}{2}(\partial \Phi)^2 - V$

(1)

- We know that the action $S = -\frac{1}{2}(\partial_x \Phi)^2 - V$ contains a domain wall solution which goes from $-v$ to $+v$. Therefore, S_E has a finite value for the DW and gives thus a contribution to the path integral:

$$\langle \Phi_F | e^{-HT_E} | \Phi_I \rangle = \int D\phi \; e^{-S_E} \neq 0$$

(2)

- This theory contains instantons that describe solutions with finite action. In the same way as a domain wall, an instanton in of winding Δn changes the winding of the configuration by Δn.

- finite action $\rightarrow$ $\langle n+\Delta n | e^{-iHT} | n \rangle \neq 0$

(3)

- From problem 1 we know that there are gauge transformations $U^{(k)}$ that change the winding number by k: $U^{(k)} |n\rangle = |n+k\rangle$

  $\rightarrow$ $|n\rangle$ is not an eigenstate of all gauge transformations.

  But the real vacuum should be gauge invariant

(4)

- The Θ-vacuum $|\Theta\rangle = \sum_{n=-\infty}^{+\infty} e^{in\Theta} |n\rangle$ describes a gauge invariant vacuum.

Check:
$$\mathcal{U}^{(k)}|\Theta\rangle = \sum_n e^{in\Theta}\,\mathcal{U}^{(k)}|n\rangle$$
$$= \sum_n e^{in\Theta}\,|n+k\rangle$$
$$= \sum_n e^{i(n-k)\Theta}\,|n\rangle$$
$$= e^{-ik\Theta}\,|\Theta\rangle$$

Comment: Θ-vacuum cannot be changed by a gauge invariant operator B. $([\mathcal{U}^{(k)}, B] = 0)$

$$0 = \langle\Theta|\,[\mathcal{U}^{(k)}, B]\,|\Theta'\rangle$$
$$= \left(e^{-im\Theta} - e^{-im\Theta'}\right)\underbrace{\langle\Theta|B|\Theta'\rangle}_{=0 \text{ for } \Theta\neq\Theta'}$$

example: Θ-vacuum does not change with time, because

$$\langle\Theta|\,H\,|\Theta'\rangle = 0$$

$\uparrow$ Hamiltonian is gauge invariant

(5)

- $$\langle\Theta'|\,e^{-iHT}\,|\Theta\rangle = \sum_{m,n} e^{-im\Theta'}\,e^{in\Theta}\,\langle m|\,e^{-iHT}\,|n\rangle$$

$$= \sum_{m,n} e^{in(\Theta-\Theta')}\,e^{-i\underbrace{(m-n)}_{=:Q}\Theta'}\,\langle m|\,e^{-iHT}\,|n\rangle$$

$$= \sum_{Q,n} e^{in(\Theta-\Theta')}\,e^{-iQ\Theta'}\,\langle n+Q|\,e^{-iHT}\,|n\rangle$$

$$= \sum_{Q,n} e^{in(\Theta-\Theta')}\,e^{-iQ\Theta'}\int DW\,\underbrace{\langle n+Q|\,e^{iS[W]}\,|n\rangle}$$

is non-zero for an instanton configuration with winding Q

only instanton configurations of winding Q give contribution to the path integral

$$= \sum_{Q,n} e^{in(\Theta-\Theta')}\,e^{-iQ\Theta'}\int DW_Q\;e^{iS[W_Q]}$$

$iS[W_Q] - in[W_Q]\Theta$

path integral $\quad \sum_{Q,n}$

$$= 2\pi\,\delta(\Theta-\Theta') \sum_Q \int DW_Q\, e^{iS[W_Q] - in[W_Q]\Theta}$$

$\sum_Q \int DW_Q = \int DW$

$$= 2\pi\,\delta(\Theta-\Theta') \int DW\, e^{iS[W] - in[W]\Theta}$$

- The action obtains through the existence of instantons an extra contribution

$$S_{new} = S - n[W]\Theta$$

where $n[W] = \frac{g^2}{8\pi^2} \int d^4x\, \mathrm{Tr}(G_{\mu\nu}\tilde{G}^{\mu\nu})$

$$\rightarrow S = \int d^4x \left(-\frac{1}{2}\mathrm{Tr}(G_{\mu\nu}G^{\mu\nu}) - \frac{\Theta g^2}{8\pi^2}\mathrm{Tr}(G_{\mu\nu}\tilde{G}^{\mu\nu})\right)$$

(6)

- parity transformation: $\vec{x} \xmapsto{P} -\vec{x}$

this implies $\vec{v} \xmapsto{P} -\vec{v}$, $\vec{a} \xmapsto{P} -\vec{a}$, $\vec{F} \longmapsto -\vec{F}$

electric force: $\vec{F} = q\vec{E} \rightarrow \vec{E} \xmapsto{P} -\vec{E}$

magnetic force: $\vec{F} = q\vec{v}\times\vec{B} \rightarrow \vec{B} \xmapsto{P} \vec{B}$

- $G^a_{\mu\nu}\tilde{G}^{\mu\nu}_a \sim \vec{E}_a \cdot \vec{B}_a \xmapsto{CP} -\vec{E}_a\cdot\vec{B}_a$

  $\rightarrow$ CP violation

- δS is not CP invariant.

  The new term leads to an electric dipole moment of the neutron.

  lattice calc.: $|d_n| \sim 3{,}6 \cdot 10^{-16}\,\Theta$ ecm

  experiment: $|d_n| < 2{,}4 \cdot 10^{-26}$ ecm

  $|\Theta| < 7 \cdot 10^{-11}$

experiment: $|d_n| < 2{,}4 \cdot 10^{-10}$ ecm

$\rightarrow |\Theta| < 7 \cdot 10^{-11}$

$\rightarrow$ strong CP problem: why this violation is so small?

$\rightarrow$ not really a problem, rather a puzzle