Topological Defects

Problem Sheet 11 15 January 2024

1. Magnetic Monopoles with Multiple Magnetic Charge

In this problem, we will show that spherically symmetric magnetic monopoles with multiple magnetic charges don't exist. The reason for this is that their energy isn't finite. In this problem, we will take again the usual $SU(2)$ gauge theory with an adjoint scalar field ϕ.

1. Argue why under infinitesimal rotation around the i-axis, the change in the scalar field and the gauge field can be written as

$$\delta_i \vec{\phi} = \varepsilon_{ijk} x_j \partial_k \vec{\phi} + \vec{\Lambda}_i \times \vec{\phi}, \tag{1}$$

$$\delta_i \vec{W}_j = \varepsilon_{imn} x_m \partial_n \vec{W}_j + \varepsilon_{ijk} \vec{W}_k + \vec{\Lambda}_i \times \vec{W}_j - \frac{1}{g} \partial_j \vec{\Lambda}_i. \tag{2}$$

   Note that a rotation acts on $\vec{\phi}$ and $\vec{W}_j$ as a combination of rotation and gauge transformation. The terms with $\vec{\Lambda}$ are coming from the gauge transformation. **Hint :** Take a look at problem sheet 9.

A field configuration in this theory is rotationally symmetric if there is a $\vec{\Lambda}$ such that the expressions (1) and (2) vanish. We fix the direction of $\vec{\phi}$ by

$$\vec{\phi} = h(r)\vec{w}, \tag{3}$$

where

$$\vec{w} \equiv \begin{pmatrix} \cos(n\varphi)\sin\theta \\ \sin(n\varphi)\sin\theta \\ \cos\theta \end{pmatrix}. \tag{4}$$

Notice that in a previous problem, you already found that the magnetic charge of such a configuration would correspond to n times the minimal charge corresponding to the 't Hooft-Polyakov magnetic monopole. By choosing the direction of $\vec{\phi}$ we fixed the gauge, but gauge transformations with $\vec{\Lambda}_i$ parallel to $\vec{\phi}$ are still possible.

2. Show that with the remaining gauge freedom, we can demand

$$x_i \vec{W}_i \cdot \vec{w} = 0 \tag{5}$$

3. Show that the rotational invariance of the two gauge constraints (3) and (5) leads to the solution

$$\vec{\Lambda}_i = \varepsilon_{ijk} x_j (\partial_k \vec{w}) \times \vec{w} + f_i(\mathbf{x})\vec{w}, \tag{6}$$

$$\frac{\partial f_i}{\partial r} = 0. \tag{7}$$

Now, let us write $\vec{W}_i = \vec{A}_i + \vec{W}_i^\infty$, with $\vec{W}_i \xrightarrow{r\to\infty} \vec{W}_i^\infty$.

4. Rotational invariance requires $\delta_i \vec{W}_j^\infty = 0$. Show that $\partial_j f_i(\mathbf{x})$ is symmetric in i and j and argue why $f_i(\mathbf{x})$ can be written as the gradient of some function $g(\mathbf{x})$, $f_i(\mathbf{x}) = \partial_i g(\mathbf{x})$.
   Hint : $\vec{W}_i^\infty$ is the magnetic monopole solution in the infinite limit (see problem sheet 8). Furthermore, you may need an identity that you already proved on problem sheet 8.
5. Argue why you can write $g(\mathbf{x}) = -nr + b_i x_i$, where b_i is some constant vector.
6. Show that rotational invariance $\delta_i \vec{W}_j = 0$ leads to the equation

$$\frac{x_i}{r}\varepsilon_{ijk}\vec{A}_k = \left(n - \frac{x_i b_i}{r}\right)\, \vec{w} \times \vec{A}_j. \tag{8}$$

7. Derive the equation

$$-\left(\frac{x_i}{r}\vec{A}_i\right)^2 = \left[\left(n - \frac{x_i b_i}{r}\right)^2 - 1\right] \vec{A}_i \cdot \vec{A}_i \tag{9}$$

   by squaring (8).
   Hint : Using part 2, explain first why $\vec{A}_i \cdot \vec{w} = 0$.
8. What is different for $|n| \geq 2$ compared to $n = 0, \pm 1$? Explain why for $|n| \geq 2$ there are solutions for $\vec{W}_i$ that have in finite regions everywhere the asymptotic value $\vec{W}_i^\infty$. What does this mean for the energy of the whole configuration ?

2. The Magnetic Monopole Problem

There were many causally disconnected regions during the phase transition where magnetic monopoles could have emerged. Let us assume that there was one magnetic monopole created per causally disconnected region of radius $r_H \sim \frac{M_{\text{Pl}}}{T_{\text{GUT}}^2}$, with $T_{\text{GUT}} \sim 10^{16}\text{GeV}$ the temperature at which the monopole of mass $m_M \sim 10^{17}\text{GeV}$ was created.

1. Assuming that the monopole was not destroyed and also no more monopoles were created until now, what would be the contribution to the energy density today ? Why is this a problem ?
   Hint : Assume that the photon density is given by $n_\gamma = \frac{2T^3}{\pi^2}\zeta(3)$ and assume that the photon number stays constant during the expansion of the universe.

In the following, we will analyze qualitatively two possible solutions to the magnetic monopole problem.

Erasure of Magnetic Monopoles
Let us consider an $SU(2)$ gauge theory with a scalar field ϕ transforming under the adjoint representation. The potential of the theory is given by

$$V(\phi) = \lambda \left(\text{Tr}\big(\phi^\dagger\phi\big) - \frac{v^2}{2} \right)^2 \text{Tr}\big(\phi^\dagger\phi\big). \tag{10}$$

2. What is a possible domain wall solution ?
3. Combine two domain walls to an $SU(2)$ invariant vacuum layer that is a layer with an $SU(2)$ invariant phase separated by two domain walls from the outer region with $U(1)$ symmetry. Remember that this vacuum layer is not a solution to the static field equations.

These vacuum layers can form during the same phase transition in which also the magnetic monopoles appear. A monopole gets erased when it encounters such a vacuum layer. In this way, the number of monopoles can be reduced to a level that is consistent with our observations of the universe.

4. Explain qualitatively why the magnetic monopole gets erased.
5. What are possible observables of the erasure mechanism ?

Monopoles connected by Strings
Langacker and Pi suggested another solution to the magnetic monopole problem. They suggested that the universe has a temporal phase in which the electromagnetic $U(1)$ symmetry is broken. This leads to the emergence of strings that connect the monopoles and pull them together. As an example model take an $SU(2)$ gauge theory. The adjoint scalar field ϕ breaks the symmetry down to $U(1)$. The complex fundamental scalar field ψ breaks the $U(1)$ symmetry.

6. Write down a potential that allows this breaking chain.
7. Think about the differences between a monopole-antimonopole pair and a monopole-antimonopole pair connected by a string. Why do monopoles connected by strings annihilate faster than those without strings ?