Topological Defects

Problem Sheet 10 — 8 January 2024

1. Homotopy Theory

In the literature arguments for the existence of topological defects are often given by the use of homotopy theory. This problem summarizes the most important terms of homotopy theory that might be useful in your career as a physicist.

Let us define the vacuum manifold of a scalar field theory more mathematically. Let ϕ_0 be one minimum of the potential and G be the symmetry group of the theory, then we call

$$\mathcal{O}_{\phi_0} = \{\phi = g\phi_0 : g \in G\}$$

the *orbit* of ϕ_0. In other words, the orbit is the set of all ϕ that are connected by a transformation in G. (Comment : For convenience, we write g but we mean the representation $T(g)$.) The vacuum manifold is defined by all field values that minimize the potential, thus

$$\mathcal{M} = \bigcup_{\phi_0 \text{ minimizes } V} \mathcal{O}_{\phi_0}.$$

1. As an example let us consider a complex scalar field theory with $U(1)$ symmetry. Lets take the potential $V_1(\phi) = \lambda(|\phi|^2 - v^2)^2$. Give an example of an orbit and write down the vacuum manifold of this theory.
2. If we replace the potential by $V_2(\phi) = \lambda(|\phi|^2 - v_1)^2(|\phi|^2 - v_2)^2$ with $v_1 \neq v_2$. How many non-equal orbits are there? What is the vacuum manifold?

The gauge group G can have elements that have no effect on the elements of the orbit. We collect these elements in the so-called *stabilizer subgroup*

$$H_{\phi_0} = \{h \in G : \phi_0 = h\phi_0\}.$$

Furthermore, let us define the *(left) coset* by

$$gH_{\phi_0} = \{gh : h \in H_{\phi_0}\}$$

and the *coset space* by

$$G/H_{\phi_0} = \{gH_{\phi_0} : g \in G\}.$$

We can observe that there is a one-to-one correspondence between G/H_{ϕ_0} and $\mathcal{O}_{\phi_0}$, i.e. each coset gH_{ϕ_0} can be mapped to one point on the orbit ϕ_0. We can combine these coset spaces into one coset space by

$$G/H = \bigsqcup_{\text{inequivalent orbits}} G/H_{\phi_0},$$

where $\bigsqcup$ denotes the disjoint union. Then there is a one-to-one correspondence between the vacuum manifold $\mathcal{M}$ and G/H.

3. What are the coset spaces G/H of the theories given above with the potentials $V_1(\phi)$ and $V_2(\phi)$? Why the disjoint union is necessary ?
4. Now consider an $SU(2)$ symmetric theory with the scalar field transforming under the adjoint representation. The potential is $V(\phi) = \lambda(\mathrm{Tr}(\phi^2) - v^2)^2$. What is the coset space ? How do the coset space and the vacuum manifold look like ?

Now that we defined the vacuum manifold mathematically, we can continue with homotopy theory. Two mappings $f : X \to Y$ and $g : X \to Y$ are called *homotopic*, if they can be continuously deformed into each other, i.e. there exists a family of continuous mappings h_t with $t \in [0, 1]$, such that $h_0 = f$ and $h_1 = g$. The set of all mappings that are homotopic to each other is called *homotopy class*.

5. Write down three non-homotopic mappings from $\mathbb{R}^2 \setminus (0, 0)$ to S_1. How many non-homotopic mappings are there ?

Let us now consider the boundary of the n-dimensional real space which can be identified by the n-sphere S_n. The group of all homotopy classes that consist of mappings from S_n to a group G is called the *homotopy group* $\pi_n(G)$.

6. What are the homotopy groups $\pi_0(Z_2)$ and $\pi_1(S_1)$?
7. What is the homotopy group $\pi_1(S_2)$?

We want to give you two important rules without proving them. First of all, you can use $\pi_n(G_1 \times G_2) = \pi_n(G_1) \times \pi_n(G_1)$. And secondly, for a compact, connected, and simply connected group G (always the case for $SU(N)$), we can write $\pi_2(G/H) = \pi_1(H)$.

8. What is the homotopy group $\pi_1(T_2)$, where T_2 describes a torrus ?
9. Consider the case with an $SU(2)$ adjoint scalar field with the potential $V(\phi) = \lambda(\mathrm{Tr}(\phi^2) - v^2)^2$. What is the homotopy group $\pi_2(G/H)$?

But why all of this is important for topological defects ? The answer is that a theory allows a domain wall, vortex, or magnetic monopole solution if the homotopy group $\pi_0(G/H)$, $\pi_1(G/H)$, or $\pi_2(G/H)$ are non-trivial, respectively. In the previous example with the adjoint scalar field, you saw that $\pi_2(G/H)$ is non-trivial, and thus a magnetic monopole solution is allowed in this theory, which we already know from the previous problem sheets.

10. Assuming there was a grand unified theory with gauge group $SU(5)$ that comprises the whole standard model, show that magnetic monopoles are allowed when the $SU(5)$ symmetry gets broken down to the standard model symmetry group.

2. Non-stable Domain Walls

In this exercise, we will analyze the domain wall solution in a $(1+1)$-dimensional complex scalar field theory with the Lagrangian

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - \lambda(|\phi|^2 - v^2)^2 \tag{1}$$

1. Derive a domain wall solution that separates a region with vacuum expectation value $\langle\phi\rangle = +v$ from a region with $\langle\phi\rangle = -v$.
2. Take perturbations $\delta\phi$ around the domain wall solution and show that they satisfy the equation

$$\partial_\mu \partial^\mu \delta\phi + 2\lambda v^2 \left(\tanh^2(\sqrt{\lambda} v x)(\delta\phi + \delta\phi^*) + \left(\tanh^2(\sqrt{\lambda} v x) - 1 \right) \delta\phi \right) = 0 \tag{2}$$

3. Take $\mathrm{Re}\, \delta\phi = 0$ and $\mathrm{Im}\, \delta\phi = \gamma(x) e^{-i\omega t}$ and show that there are solutions with ω being imaginary. What does this mean ?
4. Use the arguments from homotopy theory to explain why this is the case. Do you know what happens in the time evolution of such a domain wall ?
5. If we replace the potential by

$$V(\phi) = \lambda(|\phi|^2 - v^2)^2 |\phi|^2, \tag{3}$$

stable domain walls exist. Explain why.