Topological Defects

Problem Sheet 9 — 18 December 2023

1. The BPS Magnetic Monopole

With the knowledge of the previous problem sheet, we can write down the 't Hooft-Polyakov magnetic monopole solution. It can be written as

$$\begin{aligned} W_i^a &= \varepsilon_{aij}\frac{r^j}{r^2}\frac{1}{g}(1-K(r)) \\ W_t^a &= 0 \\ \phi^a &= \frac{r^a}{r^2}\frac{1}{g}H(r) \end{aligned} \tag{1}$$

1. *(optional)* Find an expression for the energy density in terms of $H(r)$ and $K(r)$. Which limits $K(r)$ and $H(r)$ have to satisfy at $r \to 0$ and $r \to \infty$ in order to have a finite energy solution ?
2. *(optional)* Show that the field equations for $K(r)$ and $H(r)$ are

$$H''(r) = \frac{2}{r^2}H(r)K(r)^2 + \lambda v^2 H(r)\left(\frac{H(r)^2}{m_v^2 r^2} - 1\right) \tag{2}$$

$$K''(r) = \frac{1}{r^2}\left(K(r)^3 - K(r) + H(r)^2 K(r)\right) \tag{3}$$

3. Show that in the BPS limit $\lambda \to 0$, the solution is

$$K(r) = \frac{m_v r}{\sinh(m_v r)} \tag{4}$$

$$H(r) = \frac{m_v r}{\tanh(m_v r)} - 1 \tag{5}$$

4. Show that the static energy of a 't Hooft-Polyakov monopole can be written as

$$E = \frac{4\pi v}{g} + \int \mathrm{d}^3x \left[\frac{1}{2}\left(B_i^a - (D_i\phi)^a\right)^2 + V(\phi)\right]. \tag{6}$$

5. Explain why the 't Hooft-Polyakov magnetic monopole solution in the BPS limit satisfies the equation

$$B_i^a = (D_i\phi)^a, \tag{7}$$

and minimizes the energy. What is the mass of the BPS magnetic monopole ?

2. Electromagnetic Field Strength Tensor

To obtain the electromagnetic field strength tensor, one could make the naive guess to project out the unbroken component or the gauge field, i.e. $A_\mu = \hat{\phi}^a W_\mu^a$ and use the usual expression $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

1. Show that this field strength tensor is not $SU(2)$ gauge invariant.
2. We will modify the field strength tensor by adding another term :

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \frac{1}{g}\varepsilon_{abc}\hat{\phi}^a(\partial_\mu\hat{\phi}^b)(\partial_\nu\hat{\phi}^c). \tag{8}$$

   Show that this is equivalent to

$$F_{\mu\nu} = \hat{\phi}^a G_{\mu\nu}^a - \frac{1}{g}\varepsilon_{abc}\hat{\phi}^a(D_\mu\hat{\phi})^b(D_\nu\hat{\phi})^c. \tag{9}$$

   where $G_{\mu\nu}^a$ is the $SU(2)$ field strength tensor.
3. Show that (9) is gauge invariant.
4. Show that for $\hat{\phi}^a = \delta^{a3}$, it reduces to the usual electromagnetic field strength $F_{\mu\nu} = \partial_\mu W_\nu^3 - \partial_\nu W_\mu^3$.

3. Moduli Space Metric of a Single Magnetic Monopole

The goal of this problem is to determine the moduli space metric of a single magnetic monopole. For that recall that the moduli space is described by the collective coordinates.
We have an $SU(2)$ gauge theory with an adjoint scalar field ϕ^a. For the magnetic monopole, we have to be a bit more careful than for the kink or vortex since a translation of the monopole can raise to an additional gauge transformation.

1. Explain why the scalar and vector fields change by

$$\delta\phi^a = \varepsilon_{abc}\alpha_b(x)\phi^c(x) + a_i(t)\partial_i\phi^a(x) \tag{10}$$
$$\delta W_i^a = \varepsilon_{abc}\alpha_b(x)W_i^c(x) - \frac{1}{g}\partial_i\alpha_a(x) + a_j(t)\partial_j W_i^a(x) \tag{11}$$

   if we apply the transformation $x^i \to x^i - a^i(t)$ with $a^i(t)$ being very small. Notice that we want to keep $W_t^a = 0$.
2. For $\alpha_a = g a_i(t) W_i^a(x)$ the Gauss constraint remains satisfied. Show that the kinetic term takes the form

$$T = \frac{1}{2}\dot{a}_i\dot{a}_j \int \mathrm{d}^3x \left(G_{ik}^a G_{jk}^a + (D_i\phi)^a(D_j\phi)^a\right). \tag{12}$$

3. Show that in the BPS limit, the kinetic energy can be rewritten to

$$T = \frac{1}{2}M\dot{a}_k\dot{a}_k, \tag{13}$$

   where M is the monopole mass.

In addition to the translation collective coordinates, we can have one that arises from a gauge transformation with

$$U = \exp\left(i\frac{\phi}{v}\chi(t)\right). \tag{14}$$

$\chi(t)$ can change with time and describes this fourth collective coordinates.

4. How do ϕ^a and W_i^a change with this transformation ?
5. Show that the kinetic energy corresponding to the collective coordinate χ is given by

$$T = \frac{1}{2}\frac{1}{g^2 v^2} M\dot{\chi}^2 \tag{15}$$

6. What is the moduli space metric of a single magnetic monopole ? What is the geometric form of the moduli space ?
   Hint : The moduli space metric g_{ij} is defined by $T = \frac{1}{2}g_{ij}\dot{X}_i\dot{X}_j$, where X_i are all the collective coordinates.
7. Let us assume that $\chi(t) = \lambda t$. Calculate the electric charge of the magnetic monopole (Julia-Zee Dyon).