Topological Defects

Problem Sheet 8

11 December 2023

1. Higgs Mechanism in a Non-Abelian Theory

Let us consider a scalar field theory with the following Lagrangian

$$\mathcal{L} = \mathrm{Tr}\left((\partial_\mu\phi)^\dagger(\partial^\mu\phi)\right) - \lambda\left(\mathrm{Tr}\big(\phi^\dagger\phi\big) - \frac{v^2}{2}\right)^2, \tag{1}$$

where one can write $\phi = \phi^a T^a$ with T^a being the generators of $SU(2)$ normalized by $\mathrm{Tr}\, T^a T^b = \delta^{ab}/2$. In the $SU(2)$ case $T^a = \frac{\sigma^a}{2}$, with σ^a being the Pauli matrices. The scalar field ϕ is in the adjoint representation.

1. Which geometric shape has the vacuum manifold of this theory ?
2. If we choose the vacuum expectation value

$$\langle\phi\rangle = vT^3, \tag{2}$$

   which generators are broken ? What is the remaining symmetry after symmetry breaking ?
3. Determine the mass spectrum after the breaking of the symmetry.

Now let us gauge the theory. Then the Lagrangian becomes

$$\mathcal{L} = -\frac{1}{2}\,\mathrm{Tr}\left(G_{\mu\nu}G^{\mu\nu}\right) + \mathrm{Tr}\left((D_\mu\phi)^\dagger(D^\mu\phi)\right) - \lambda\left(\mathrm{Tr}\big(\phi^\dagger\phi\big) - \frac{v^2}{2}\right)^2, \tag{3}$$

where $G_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu - ig[W_\mu, W_\nu]$ and $D_\mu\phi = \partial_\mu\phi - ig[W_\mu, \phi]$.

4. Take the same vacuum expectation value as above and determine the mass spectrum for the gauge bosons.

2. Winding Number for $SU(2)$ in 3 Dimensions

For an $SU(2)$ field theory with adjoint scalar field ϕ, we can define the topological current by

$$J^\mu = -\frac{1}{8\pi}\varepsilon^{\mu\alpha\beta\gamma}\varepsilon_{abc}\partial_\alpha\hat{\phi}^a\partial_\beta\hat{\phi}^b\partial_\gamma\hat{\phi}^c. \tag{4}$$

We take the convention $\varepsilon_{0123} = +1$.

1. Show that this current is conserved, i.e. $\partial_\mu J^\mu = 0$.

The topological charge (winding number) of the ϕ configuration is described by $Q \equiv \int \mathrm{d}^3x J^0$. Let us take

$$\hat{\phi} = \hat{w} \equiv \begin{pmatrix} \cos(n\varphi)\sin\theta \\ \sin(n\varphi)\sin\theta \\ \cos\theta \end{pmatrix} \tag{5}$$

2. Convince yourself that the following relation holds

$$(\partial_i \hat{w} \times \partial_j \hat{w}) \cdot \hat{w} = n\varepsilon_{ijk}\frac{r^k}{r^3}. \tag{6}$$

   Comment : The calculations are straightforward but are very tedious. You can use Mathematica.

3. Show that the winding number is $Q = n$.

3. 't Hooft-Polyakov Magnetic Monopole

In this problem, we want to find the magnetic monopole solution in the $SU(2)$ gauge theory given by the Lagrangian in equation (3).

1. Find the energy density for static field configurations for $W_t = 0$.
2. In order to obtain a static finite energy solution, the potential has to vanish at infinity, i.e. $|\vec{\phi}| \xrightarrow{r\to\infty} v$. This condition is satisfied with $\vec{\phi} \xrightarrow{r\to\infty} v\hat{w}$. Furthermore, we have to require $D_i\phi \xrightarrow{r\to\infty} 0$. Using this, show that the solution for the gauge field

$$\vec{W}_i \xrightarrow{r\to\infty} \frac{1}{g}(\partial_i \hat{w}) \times \hat{w} + c_i \hat{w}, \tag{7}$$

   satisfies this condition, where c_i is a vector that can be coordinate-dependent.
3. In the $r \to \infty$ limit, compute the magnetic field $B_k = -\frac{1}{2}\varepsilon_{kij}F_{ij}$, where $F_{\mu\nu}$ is the electromagnetic field strength tensor that can be obtained by projecting out the unbroken $SU(2)$ direction of $G_{\mu\nu}$. For doing that you can use the scalar product $\langle A, B\rangle = 2\,\mathrm{Tr}\,AB$, where A and B are arbitrary matrices 2×2 matrices.
4. We want to find a solution that describes a magnetic monopole. Therefore, the magnetic field should go as $B_k \xrightarrow{r\to\infty} r_k/r^3$. Can you set $c_i = 0$ by using the gauge freedom ?
5. What is the charge of the magnetic monopole for $n = 1$? Notice that it seems that you can have magnetic monopoles with higher charges, due to the number n. However, in another problem, we will show that these solutions have no finite energy.