07 Problem Sheet

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Problem 1:

(1)

- $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad \text{with} \quad |h_{\mu\nu}| \ll 1$

- $$\Gamma^\mu_{\alpha\beta} = \frac{1}{2} g^{\mu\nu}\left(g_{\nu\alpha,\beta} + g_{\nu\beta,\alpha} - g_{\alpha\beta,\nu}\right)$$
$$\approx \frac{1}{2}\eta^{\mu\nu}\left(h_{\nu\alpha,\beta} + h_{\nu\beta,\alpha} - h_{\alpha\beta,\nu}\right)$$

- $$R^\alpha{}_{\beta\mu\nu} = \partial_\mu \Gamma^\alpha_{\nu\beta} - \partial_\nu \Gamma^\alpha_{\mu\beta} + \underbrace{\cancel{\Gamma^\alpha_{\mu\lambda}\Gamma^\lambda_{\nu\beta}}}_{\sim \mathcal{O}(h^2)} - \underbrace{\cancel{\Gamma^\alpha_{\nu\lambda}\Gamma^\lambda_{\mu\beta}}}_{\sim \mathcal{O}(h^2)}$$
$$= \frac{1}{2}\eta^{\alpha\lambda}\left(h_{\lambda\nu,\beta\mu} + \cancel{h_{\lambda\beta,\nu\mu}} - h_{\nu\beta,\lambda\mu} - h_{\lambda\mu,\beta\nu} - \cancel{h_{\lambda\beta,\mu\nu}} + h_{\mu\beta,\lambda\nu}\right)$$

- $$R_{\beta\nu} = R^\alpha{}_{\beta\alpha\nu} = \frac{1}{2}\left(h^\alpha{}_{\nu,\beta\alpha} - h_{\nu\beta,}{}^\alpha{}_\alpha - h^\alpha{}_{\alpha,\beta\nu} + h^\alpha{}_{\beta,\alpha\nu}\right)$$

Using the harmonic gauge $\partial_\mu h^\mu{}_\nu = \frac{1}{2}\partial_\nu \underset{h^\alpha{}_\alpha}{\underset{\shortparallel}{h}}$

$$R_{\beta\nu} = \frac{1}{2}\left(\frac{1}{2}\partial_\beta\partial_\nu h - \Box h_{\beta\nu} - \partial_\beta\partial_\nu h + \frac{1}{2}\partial_\beta\partial_\nu h\right)$$
$$= -\frac{1}{2}\Box h_{\beta\nu}$$

- With this the Einstein equation $R_{\mu\nu} = 8\pi G\left(T_{\mu\nu} - \frac{1}{2}T\eta_{\mu\nu}\right)$ becomes

$$\Box h_{\mu\nu} = -16\pi G\left(T_{\mu\nu} - \frac{1}{2}T\eta_{\mu\nu}\right)$$

(2)

- $$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\Phi)^*(D^\mu\Phi) - V(\Phi)$$
$$= -\frac{1}{4}g^{\mu\alpha}g^{\nu\beta}F_{\mu\nu}F_{\alpha\beta} + g^{\mu\nu}(D_\mu\Phi)^*(D_\nu\Phi) - V(\Phi)$$

- Hilbert energy momentum tensor

$$T_{\mu\nu} = 2\frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} - g_{\mu\nu}\mathcal{L}$$

$$= F_{\mu\alpha}F^{\alpha}{}_{\nu} + \frac{1}{4}g_{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}$$
$$+ (D_\mu\Phi)^*(D_\nu\Phi) + (D_\nu\Phi)^*(D_\mu\Phi) - g_{\mu\nu}(D_\alpha\Phi)^*(D^\alpha\Phi) + g_{\mu\nu}V(\Phi)$$

- Because of symmetry and the reason that the configuration is static we can argue that

$$A_0 = 0 \quad , \quad A_3 = 0$$
$$\partial_z A_\mu = 0 \quad , \quad \partial_z\Phi = 0$$

$$\rightarrow F_{0\alpha} = 0 \quad , \quad F_{3\alpha} = 0$$
$$D_0\Phi = \partial_t\Phi + igA_0\Phi = 0$$
$$D_3\Phi = 0$$

- $$T_{00} = \underbrace{F_{0i}F^{i}{}_{0}}_{=0} + \frac{1}{4}F_{ij}F_{ij} + \underbrace{2(D_0\Phi)^*(D_0\Phi)}_{=0} + (D_i\Phi)^*(D_i\Phi) + V(\Phi)$$
$$= -\mathcal{L} = \mathcal{E}$$

The other searched components can be found similarly

$$T_{33} = \mathcal{L} = -\mathcal{E}$$
$$T_{0i} = T_{i0} = 0$$
$$T_{3i} = T_{i3} = 0$$

Comment: In the BPS limit the other components are also zero. In the next part we are also zero. In the next part we will use an approximation that is also valid besides the BPS limit.

(3)

- $\tilde{T}_{\mu\nu}(x,y) = \delta(x)\delta(y) \int T_{\mu\nu}(x,y)\, dx\, dy$

$$\partial_\mu T^{\mu\nu} = 0$$

$$\rightarrow 0 = \int x^k \partial_j T^{ji}\, dx\, dy$$

$$= \int x^k (\partial_x T^{1i} + \partial_y T^{2i})\, dx\, dy$$

$$= \int dy \left(\int x^k \partial_x T^{1i} dx \right) + \int dx \left(\int x^k \partial_y T^{2i} dy \right)$$

$$= \int dy \left(\underbrace{x^k T^{1i} \Big|_{x \to \pm\infty}}_{=0} - \int \delta_{1k} T^{1i} dx \right)$$

$$+ \int dx \left(\underbrace{x^k T^{2i} \Big|_{y \to \pm\infty}}_{=0} - \int \delta_{2k} T^{2i} dy \right)$$

Note that $T^{\mu\nu}$ decays faster than $\frac{1}{r}$, otherwise the integral is divergent.

$$\rightarrow \int T^{ij}\, dx\, dy = 0 \qquad \text{for} \quad i,j \in \{1,2\}$$

$$\Rightarrow \tilde{T}_{\mu\nu} = \varepsilon\, \delta(x)\delta(y)\, \mathrm{diag}(1,0,0,-1)$$

(4)

- $T = T^\mu{}_\mu = 2\varepsilon\, \delta(x)\delta(y)$

$$\rightarrow \Box h_{\mu\nu} = -16\pi G \varepsilon \left(\mathrm{diag}(1,0,0,-1) - \mathrm{diag}(1,-1,-1,-1) \right) \delta(x)\delta(y)$$

$$= -16\pi G \varepsilon\, \mathrm{diag}(0,1,1,0)\, \delta(x)\delta(y)$$

$$(\partial_x^2 + \partial_y^2)\, h_{\mu\nu} = 16\pi G \mathcal{E}\ \mathrm{diag}(0,1,1,0)\, \delta(x)\delta(y)$$

- Transforming into cylindrical coordinates

$$\delta(\vec{f}(\vec{x})) = \underbrace{\frac{1}{|\det \vec{f}\,'|}}_{\text{Jacobian}}\ \delta(\vec{x})$$

$$\rightarrow \delta(x)\,\delta(y) = \frac{1}{r}\,\delta(r)\,\delta(\varphi)$$

$$\partial_x^2 + \partial_y^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}$$

- $h_{\mu\nu}$ is cylindrically symmetric, i.e. it doesn't depend on φ.

$$\rightarrow \frac{\partial}{\partial r}\left(r\frac{\partial h_{\mu\nu}}{\partial r}\right) = 16\pi G\mathcal{E}\ \mathrm{diag}(0,1,1,0)\ \delta(r)\delta(\varphi)$$

- Integration over φ from 0 to 2π:

$$2\pi\frac{\partial}{\partial r}\left(r\frac{\partial h_{\mu\nu}}{\partial r}\right) = 16\pi G\mathcal{E}\ \mathrm{diag}(0,1,1,0)\ \delta(r)$$

Integration over r from 0 to ∞:

$$r\cdot\frac{\partial h_{\mu\nu}}{\partial r} = 8G\mathcal{E}\ \mathrm{diag}(0,1,1,0)$$

Integration over r:

$$h_{11} = h_{22} = 8G\mathcal{E}\ \ln\frac{r}{r_0}$$

integration const.

(5)

- $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$

$$= \mathrm{diag}\left(1,\ -1 + 8G\mathcal{E}\ln\frac{r}{r_0},\ -1 + 8G\mathcal{E}\ln\frac{r}{r_0},\ -1\right)$$

$$= \mathrm{diag}\left(1,\ -1+8G\varepsilon \ln\frac{r}{r_0},\ -1+8G\varepsilon\ln\frac{r}{r_0},\ -1\right)$$

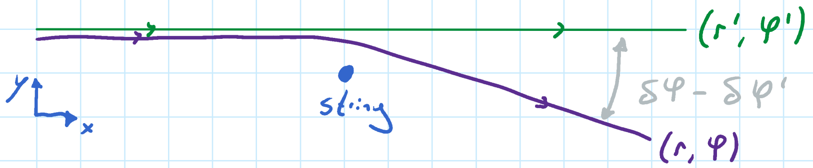
$$ds^2 = dt^2 - dz^2 - \left(1-8G\varepsilon\ln\frac{r}{r_0}\right)\left(dr^2+r^2d\varphi^2\right)$$
$$= dt^2 - dz^2 - dr'^2 - r'^2 d\varphi'^2$$

where $\left(1-8G\varepsilon\ln\frac{r}{r_0}\right)r^2 = (1-8G\varepsilon)r'^2$

$\varphi' = (1-4G\varepsilon)\varphi$

- In the (r',φ') coordinates, the trajectories are straight lines. But not in the (r,φ) coordinates, because $\varphi' = (1-4G\varepsilon)\varphi$.
  If $\delta\varphi' = \pi$, then $\delta\varphi = \frac{\pi}{1-4G\varepsilon} > \pi$

- sketch:



Problem 2:

(1)

- $D_\mu\Phi = \partial_\mu\Phi - ig\left[W_\mu,\Phi\right]$

$$\longmapsto \partial_\mu\left(U\Phi U^+\right) - ig\left[W'_\mu,\ U\Phi U^+\right]$$

$$\left(\partial_\mu U\right)\Phi U^+ + U\left(\partial_\mu\Phi\right)U^+ + U\Phi\left(\partial_\mu U^+\right)$$

$$-ig\,W'_\mu U\Phi U^+ + ig\,U\Phi U^+ W'_\mu$$

$$= U\left(D_\mu\Phi\right)U^+ + \left(\partial_\mu U\right)\Phi U^+ + U\Phi\left(\partial_\mu U^+\right)$$
$$- ig\,W'_\mu U\Phi U^+ + ig\,U\Phi U^+ W'_\mu$$

$$- ig\, W_\mu' U\phi U^+ + ig\, U\phi U^+ W_\mu'$$
$$+ ig\, U W_\mu \phi U^+ - ig\, U\phi W_\mu U^+$$

$$= U(D_\mu \phi)U^+ \quad \text{for} \quad W_\mu' = U W_\mu U^+ - \frac{i}{g}(\partial_\mu U)U^+$$

$$\rightarrow W_\mu \longmapsto U W_\mu U^+ + \frac{i}{g} U(\partial_\mu U^+)$$

(2)

$$\bullet\ G_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu - ig[W_\mu, W_\nu]$$

$$\longmapsto \partial_\mu\left(U W_\nu U^+ + \frac{i}{g}U(\partial_\nu U^+)\right) - \partial_\nu\left(U W_\mu U^+ + \frac{i}{g}U(\partial_\mu U^+)\right)$$
$$- ig\left[U W_\mu U^+ + \frac{i}{g}U(\partial_\mu U^+),\ U W_\nu U^+ + \frac{i}{g}U(\partial_\nu U^+)\right]$$
$$= (\partial_\mu U)W_\nu U^+ + U(\partial_\mu W_\nu)U^+ + U W_\nu(\partial_\mu U^+)$$
$$+ \frac{i}{g}(\partial_\mu \cancel{U)}(\partial_\nu U^+) + \frac{i}{g}U\cancel{\partial_\mu}\partial_\nu U^+$$
$$- (\partial_\nu U)W_\mu U^+ - U(\partial_\nu W_\mu)U^+ - U W_\mu(\partial_\nu U^+)$$
$$- \frac{i}{g}(\partial_\nu \cancel{U)}(\partial_\mu U^+) - \frac{i}{g}U\cancel{\partial_\nu}\partial_\mu U^+$$
$$- ig\, U[W_\mu, W_\nu]U^+ + [U(\partial_\mu U^+), U W_\nu U^+]$$
$$+ [U W_\mu U^+, U(\partial_\nu U^+)] + \frac{i}{g}\underbrace{[U(\partial_\mu U^+), U(\partial_\nu U^+)]}_{= -(\partial_\mu \cancel{U)}(\partial_\nu U^+) + (\partial_\nu \cancel{U)}(\partial_\mu U^+)}$$

$$= U G_{\mu\nu} U^+ + (\partial_\mu \cancel{U)W_\nu} U^+ + U \cancel{W_\nu} (\partial_\mu U^+)$$
$$- (\partial_\nu \cancel{U)W_\mu} U^+ - U \cancel{W_\mu} (\partial_\nu U^+)$$
$$+ U(\partial_\mu \cancel{U^+)}U W_\nu U^+ - U \cancel{W_\nu} (\partial_\mu U^+)$$
$$+ U W_\mu \cancel{(\partial_\nu} U^+) - U(\partial_\nu \cancel{U^+)}U W_\mu U^+$$

$$= U G_{\mu\nu} U^+$$

- $\mathrm{Tr}(G_{\mu\nu}G^{\mu\nu}) \longmapsto \mathrm{Tr}(U G_{\mu\nu} U^\dagger U G^{\mu\nu} U^\dagger)$

$$= \mathrm{Tr}(G_{\mu\nu}G^{\mu\nu}U^\dagger U) = \mathrm{Tr}(G_{\mu\nu}G^{\mu\nu})$$

Similarily, $\mathrm{Tr}((D_\mu\Phi)^\dagger(D^\mu\Phi))$ invariant

$\Rightarrow \mathcal{L}$ is invariant under $SU(N)$ gauge transformation

(3)

- $U = e^{i\varphi^a T^a}$

$$U\Phi U^\dagger = e^{i\varphi^a T^a}\,\Phi^b T^b\, e^{-i\varphi^c T^c}$$

$$\approx (1 + i\varphi^a T^a)\,\Phi^b T^b\,(1 - i\varphi^c T^c)$$

$$\approx \Phi^b T^b + i\varphi^a \Phi^b T^a T^b - i\varphi^c \Phi^b T^b T^c$$

$$= \Phi^a T^a + i\varphi^a \Phi^b [T^a, T^b]$$

$$= \Phi^a T^a - f_{abc}\,\varphi^a \Phi^b T^c$$

$\rightarrow \Phi^a \longmapsto \Phi^a - f_{abc}\,\varphi^b \Phi^c$

- $U W_\mu U^\dagger + \frac{i}{g} U \partial_\mu U^\dagger$

$$= W_\mu^a T^a - f_{abc}\,\varphi^a W_\mu^b T^c + \frac{i}{g} U U^\dagger \cdot (-i)(\partial_\mu \varphi^a) T^a$$

$\rightarrow W_\mu^a \longmapsto W_\mu^a - f_{abc}\,\varphi^b W_\mu^c + \frac{1}{g}\partial_\mu \varphi^a$

(4)

- $G_{\mu\nu} = \partial_\mu W_\nu^a T^a - \partial_\nu W_\mu^a T^a - ig[W_\mu^a T^a, W_\nu^b T^b]$

$$= T^a \partial_\mu W_\nu^a - T^a \partial_\nu W_\mu^a + g f_{abc} W_\mu^a W_\nu^b T^c$$

$$= G_{\mu\nu}^a T^a$$

with $G_{\mu\nu}^a = \partial_\mu W_\nu^a - \partial_\nu W_\mu^a + g f_{abc} W_\mu^b W_\nu^c$

with $G^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g f_{abc} W^b_\mu W^c_\nu$

- $D_\mu \Phi = \partial_\mu \Phi^a T^a - ig [W^a_\mu T^a, \Phi^b T^b]$
  $= T^a \partial_\mu \Phi^a + g f_{abc} W^a_\mu \Phi^b T^c$
  $= (D_\mu \Phi)^a T^a$

with $(D_\mu \Phi)^a = \partial_\mu \Phi^a + g f_{abc} W^b_\mu \Phi^c$

- $\mathcal{L} = -\frac{1}{2} \mathrm{Tr}(G_{\mu\nu} G^{\mu\nu}) + \mathrm{Tr}((D_\mu \Phi)^\dagger (D^\mu \Phi))$
  $= -\frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a + \frac{1}{2} (D_\mu \Phi)^a (D^\mu \Phi)^a$

- $\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^a)} \right) - \frac{\partial \mathcal{L}}{\partial \Phi^a} = 0$

$\partial_\mu (D^\mu \Phi)^a - (D_\mu \Phi)^b g f_{abc} W^\mu_c = 0$

$\rightarrow (D_\mu D^\mu \Phi)^a = 0$

- $\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu W^a_\nu)} \right) - \frac{\partial \mathcal{L}}{\partial W^a_\nu} = 0$

$- \partial_\mu G^{\mu\nu}_a + G^c_{\mu\nu} g f_{cab} W^b_\mu - (D^\nu \Phi)^c g f_{cab} \Phi^b$

$\rightarrow (D_\mu G^{\mu\nu})^a = - g f_{abc} \Phi^b (D^\nu \Phi)^c$

(5)

- $\Phi^a \longmapsto \Phi^a \underbrace{- f_{abc} \varphi^b \Phi^c}_{= \delta \Phi^a}$

$W^a_\mu \longmapsto W^a_\mu \underbrace{- f_{abc} \varphi^b W^c_\mu + \frac{1}{g} \partial_\mu \varphi^a}$

const.

$$W_\mu^a \longmapsto W_\mu^a \underbrace{- f_{abc}\, \varphi^b W_\mu^c + \frac{1}{g} \partial_\mu \varphi^a}_{= \delta W_\mu^a}$$

- $$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^a)} \delta \Phi^a + \frac{\partial \mathcal{L}}{\partial(\partial_\mu W_\nu^a)} \delta W_\nu^a$$

$$= (D^\mu \Phi)^a \delta \Phi^a - G_a^{\mu\nu} \delta W_\nu^a$$

$$= - f_{abc}\, \varphi^b \Phi^c (D^\mu \Phi)^a + f_{abc}\, \varphi^b W_\nu^c G_a^{\mu\nu}$$

$$\rightarrow J_a^\mu = f_{abc} W_\nu^b G_c^{\mu\nu} - f_{abc} \Phi^b (D^\mu \Phi)^c$$

(6)

- Jacobi identity:

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$$

- $$[D_\mu, D_\nu]\Phi = D_\mu(\partial_\nu \Phi - ig[W_\nu, \Phi]) - D_\nu(\partial_\mu \Phi - ig[W_\mu, \Phi])$$

$$= \underline{-ig[W_\mu, \partial_\nu \Phi]} \underline{- ig\, \partial_\mu [W_\nu, \Phi]} - g^2 [W_\mu, [W_\nu, \Phi]]$$

$$\underline{+ ig[W_\nu, \partial_\mu \Phi]} \underline{+ ig\, \partial_\nu [W_\mu, \Phi]} + g^2 [W_\nu, [W_\mu, \Phi]]$$

$$= + ig[\partial_\nu W_\mu, \Phi] - ig[\partial_\mu W_\nu, \Phi]$$

$$- g^2 [\Phi, [W_\nu, W_\mu]]$$

$$= -ig[\partial_\mu W_\nu - \partial_\nu W_\mu - ig[W_\mu, W_\nu], \Phi]$$

$$= -ig[G_{\mu\nu}, \Phi]$$

- $$[D_\alpha, [D_\mu, D_\nu]]\Phi$$

$$= -ig\, D_\alpha([G_{\mu\nu}, \Phi]) - [D_\mu, D_\nu](D_\alpha \Phi)$$

$$= -ig[D_\alpha G_{\mu\nu}, \Phi] - ig[G_{\mu\nu}, D_\alpha \Phi] + ig[G_{\mu\nu}, D_\alpha \Phi]$$

$$= -ig[D_\alpha G_{\mu\nu}, \Phi]$$

$$= -ig\,[\underbrace{D_\alpha G_{\mu\nu}, \Phi}_{\neq 0}]$$

- $0 = [D_\alpha, [D_\mu, D_\nu]]\Phi + [D_\mu, [D_\nu, D_\alpha]]\Phi + [D_\nu, [D_\alpha, D_\mu]]\Phi$

$$= -ig\,[D_\alpha G_{\mu\nu} + D_\mu G_{\nu\alpha} + D_\nu G_{\alpha\mu}, \Phi]$$

$$\rightarrow \varepsilon^{\alpha\beta\mu\nu} D_\beta G_{\mu\nu} = 0$$

(7)

- $W_\mu = \frac{i}{g} U \partial_\mu U^\dagger$

$$\rightarrow G_{\mu\nu} = \frac{i}{g}\partial_\mu (U\partial_\nu U^\dagger) - \frac{i}{g}\partial_\nu (U\partial_\mu U^\dagger) + \frac{i}{g}[U\partial_\mu U^\dagger, U\partial_\nu U^\dagger]$$

$$= \frac{i}{g}(\partial_\mu \cancel{U)(}\partial_\nu U^\dagger) + \frac{i}{g}U \cancel{\partial_\mu \partial_\nu} U^\dagger$$

$$- \frac{i}{g}(\partial_\nu \cancel{U)(}\partial_\mu U^\dagger) - \frac{i}{g}U \cancel{\partial_\nu \partial_\mu} U^\dagger$$

$$+ \frac{i}{g}\underbrace{U(\cancel{\partial_\mu U^\dagger})}_{= -(\partial_\mu U)U^\dagger} U(\partial_\nu U^\dagger) - \frac{i}{g}U(\cancel{\partial_\nu U^\dagger})U(\partial_\mu U^\dagger)$$

$$= 0$$

$$\Rightarrow \mathrm{Tr}(G_{\mu\nu}G^{\mu\nu}) = 0$$

- Since we already know that $G_{\mu\nu}$ transforms as $G_{\mu\nu} \longmapsto U G_{\mu\nu} U^\dagger$, there is a way easier way to show this.

For $W_\mu = 0$, $G_{\mu\nu} = 0$.

Now if you apply the gauge transformation

$$W_\mu = 0 \longmapsto 0 + \frac{i}{g} U \partial_\mu U^\dagger$$

$G_{\mu\nu}$ will still remain zero.

Therefore, for a total gauge, $G_{\mu\nu} = 0$.