Problem 1:

(1)

- $$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^4}{\lambda}\left(1-\cos\left(\frac{\sqrt{\lambda}}{m}\phi\right)\right)$$

Taylor

$$\approx \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^4}{\lambda}\left(1-1+\frac{\lambda}{2m^2}\phi^2 - \frac{\lambda^2}{24m^4}\phi^4\right)$$

$$= \frac{1}{2}\underset{1}{\partial_\mu}\underset{0}{\phi}\underset{1}{\partial^\mu}\underset{0}{\phi} - \frac{1}{2}m^2\phi^2 + \frac{\lambda}{24}\phi^4$$

$\rightarrow [m] = 1 \quad , \quad [\lambda] = 2$

mass of ϕ is m

(2)

- field equations:

$$\partial_\mu\partial^\mu\phi + \frac{m^3}{\sqrt{\lambda}}\sin\left(\frac{\sqrt{\lambda}}{m}\phi\right) = 0$$

$$\tilde{\phi} := \frac{\sqrt{\lambda}}{m}\phi$$

$$\rightarrow \partial_\mu\partial^\mu\tilde{\phi} - m^2\sin\tilde{\phi} = 0 \qquad \text{with} \quad \tilde{t} = mt \;,\; \tilde{x} = mx$$

$$\partial_{\tilde{t}}^2\tilde{\phi} - \partial_{\tilde{x}}^2\tilde{\phi} + \sin\tilde{\phi} = 0$$

- solution: $\tilde{\phi} = 4\arctan\left(u\,\frac{\sinh(\gamma\tilde{x})}{\cosh(\gamma u\tilde{t})}\right)$

Check in Mathematica.

(3)

- see Mathematica:

This solution describes two colliding DWs.

They bounce off each other.

(4)

- $$\partial_t \Phi = -\frac{4m^2u^2\gamma}{\sqrt{\lambda}} \left(1+u^2 \frac{\sinh^2(\gamma m x)}{\cosh^2(\gamma m u t)}\right)^{-1} \frac{\sinh(\gamma m x)\sinh(\gamma m u t)}{\cosh^2(\gamma m u t)}$$

$$\partial_x \Phi = \frac{4m^2u\gamma}{\sqrt{\lambda}} \left(1+u^2 \frac{\sinh^2(\gamma m x)}{\cosh^2(\gamma m u t)}\right)^{-1} \frac{\cosh(\gamma m x)}{\cosh(\gamma m u t)}$$

- Since the energy is conserved we can calculate it at one specific time point $(t=0)$:

$$\partial_t \Phi \xrightarrow{t\to 0} 0$$

$$\partial_x \Phi \xrightarrow{t\to 0} \frac{4m^2u\gamma}{\sqrt{\lambda}} \frac{\cosh(\gamma m x)}{1+u^2\sinh^2(\gamma m x)}$$

$$V(\Phi) = \frac{m^4}{\lambda}\left[1-\cos\left(\frac{\sqrt{\lambda}}{m}\Phi\right)\right]$$

$$\xrightarrow{t\to 0} \frac{m^4}{\lambda}\left[1-\cos\left(4\arctan\left(u\sinh(\gamma m x)\right)\right)\right]$$

- $$E = \int_{-\infty}^{\infty} dx \left[\frac{1}{2}(\partial_t\Phi)^2 + \frac{1}{2}(\partial_x\Phi)^2 + V(\Phi)\right]$$

$$= \int_{-\infty}^{\infty} dx \left[\frac{1}{2}\left(\partial_x \Phi\big|_{t=0}\right)^2 + V(\Phi)\big|_{t=0}\right]$$

$$= \int_{-\infty}^{\infty} dx \left[\frac{8m^4u^2\gamma^2}{\lambda}\frac{\cosh^2(m\gamma x)}{(1+u^2\sinh^2(m\gamma x))^2} + \frac{2m^4}{\lambda}\sin^2\left(2\arctan\left(u\cdot\sinh(m\gamma x)\right)\right)\right]$$

$\tilde{x} = m\gamma x$

$$= \int_{-\infty}^{\infty} d\tilde{x}\left[\frac{8m^3u^2\gamma}{\lambda}\frac{\cosh^2(\tilde{x})}{(1+u^2\sinh^2(\tilde{x}))^2} + \frac{2m^3}{\lambda\gamma}\sin^2\left(2\arctan\left(u\cdot\sinh(\tilde{x})\right)\right)\right]$$

Mathematica

$$= \frac{8m^3u^2\gamma}{\lambda}\left(\frac{\arctan(\sqrt{u^2-1})}{\sqrt{u^2-1}} + \frac{1}{u^2}\right)$$

$$+ \frac{2m^3}{\lambda\gamma}\left(-4u^2\gamma^3\left(\operatorname{arctanh}(\gamma(1-iu)) + \operatorname{arctanh}(\gamma(1+iu))\right) + 4\gamma^2\right)$$

- Identity: $\operatorname{arctanh}(x) = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right)$

$$\frac{\arctan\left(\sqrt{u^2-1}\right)}{\sqrt{u^2-1}} = -i\gamma \arctan\left(i\,\frac{1}{\gamma}\right)$$
$$= \gamma \operatorname{arctanh}\left(\frac{1}{\gamma}\right)$$
$$= \frac{1}{2}\gamma \ln\left(\frac{1+\frac{1}{\gamma}}{1-\frac{1}{\gamma}}\right)$$
$$= \frac{1}{2}\gamma \ln\left(\frac{\gamma+1}{\gamma-1}\right)$$

- $\operatorname{arctanh}\left(\frac{1-iu}{\sqrt{1-u^2}}\right) + \operatorname{arctanh}\left(\frac{1+iu}{\sqrt{1-u^2}}\right)$

$$= \frac{1}{2} \ln\left(\frac{1+\gamma(1-iu)}{1-\gamma(1-iu)}\right) + \frac{1}{2} \ln\left(\frac{1+\gamma(1+iu)}{1-\gamma(1+iu)}\right)$$
$$= \frac{1}{2} \ln\left(\frac{1+\gamma(1-iu)+\gamma(1+iu)+\gamma^2(1+u^2)}{1-\gamma(1-iu)-\gamma(1+iu)+\gamma^2(1+u^2)}\right)$$
$$= \frac{1}{2} \ln\left(\frac{1+2\gamma+\gamma^2(1+u^2)}{1-2\gamma+\gamma^2(1+u^2)}\right) \quad \text{with} \quad 1+\frac{1+u^2}{1-u^2} = 2\gamma^2$$
$$= \frac{1}{2} \ln\left(\frac{2\gamma^2+2\gamma}{2\gamma^2-2\gamma}\right) = \frac{1}{2} \ln\left(\frac{\gamma+1}{\gamma-1}\right)$$

- $E = \frac{8m^3u^2\gamma}{\lambda}\left(\frac{1}{2}\cancel{\gamma \ln}\left(\frac{\gamma+1}{\gamma-1}\right) + \frac{1}{u^2}\right)$

$$+ \frac{2m^3}{\lambda\gamma}\left(-4u^2\cancel{\gamma^3\left(\frac{1}{2}\ln\left(\frac{\gamma+1}{\gamma-1}\right)\right)} + 4\gamma^2\right)$$
$$= \frac{8m^3\gamma}{\lambda} + \frac{8m^3\gamma}{\lambda} = \frac{16m^3\gamma}{\lambda} = 2M_{DW}\gamma$$

(5)

- $\tilde{\Phi} = 4\arctan\left(\frac{1}{u}\,\frac{\sinh(\gamma u \tilde{t})}{\cosh(\gamma\tilde{x})}\right)$

see Mathematica:

This solution describes a DW and an anti-DW that collide and go through each other

(6)

- $\tilde{\Phi} = 4\arctan\left(\frac{-i}{s} \frac{\sinh(\gamma i s \tilde{t})}{\cosh(\gamma \tilde{x})}\right)$ where $\gamma = \frac{1}{\sqrt{1+s^2}}$

$= 4\arctan\left(\frac{1}{s} \frac{\sin(\gamma s \tilde{t})}{\cosh(\gamma \tilde{x})}\right)$

- see Mathematica:

This solution describes two bounded DW's that are oscillating.

Problem 2:

- Moduli Apporximation:

We consider solitons as objects described by collective coordinates that can change slowly.

(1)

- The kink in 1+1 dimensions has a single collective coordinate, its position:

$$\Phi_{DW}(x - \underbrace{a(t)}_{\text{position}})$$

- Kinetic energy:

$$T = \int dx \ \frac{1}{2}\left(\partial_t \Phi_{DW}(x-a(t))\right)^2$$

$$= \int dx \ \frac{1}{2} \ \Phi_{DW}'(x)^2 \ \dot{a}(t)^2$$ with Bogomolny equation

$$= \int dx \; \frac{1}{2} \; \Phi_{DW}'(x)^2 \; \dot{a}(t)^2 \qquad \text{with Bogomolny equation}$$

$$= \frac{1}{2} \int dx \left(\frac{1}{2} \Phi_{DW}'^2(x) + V(\Phi_{DW}) \right) \dot{a}(t)^2$$

$$= \frac{1}{2} \mathcal{M} \; \dot{a}(t)^2$$

(2)

- Field equations:

$$-\partial_\mu F^{\mu\nu} + ig\Phi^* D^\nu \Phi - ig\Phi(D^\nu\Phi)^* = 0$$

$$\nu = i: \qquad \partial_\mu F^{\mu i} + ig\Phi^* D_i \Phi - ig\Phi(D_i\Phi)^* = 0 \qquad (1)$$

$$\nu = 0: \qquad \partial_i F_{i0} + ig\Phi^* D_0 \Phi - ig\Phi(D_0\Phi)^* = 0 \qquad \text{(Gauss contraint)}$$

- If we insert the static vortex solution we have

$$F_{i0} = \partial_i A_0 - \partial_t A_i = 0$$

$$D_0 \Phi = \partial_t \Phi + ig A_0 \Phi = 0$$

Thus, the Gauss contraint is satisfied.

When we apply the transformation, we get

$$F_{i0} = \partial_i A_0 - \partial_t A_i$$

$$\longmapsto \underbrace{\partial_i A_0 - \partial_t A_i}_{\substack{=0 \text{ for static} \\ \text{vortex}}} + \underbrace{\dot{a}_j \partial_j A_i}_{\neq 0} + a_j \partial_j \underbrace{\partial_t A_i}_{\substack{=0 \text{ for static} \\ \text{vortex}}}$$

$$D_0 \Phi = \partial_t \Phi + ig A_0 \Phi$$

$$\longmapsto \underbrace{\partial_t \Phi}_{\substack{=0 \text{ for} \\ \text{static vortex}}} - \dot{a}_i \partial_i \Phi + ig \underbrace{A_0}_{\substack{=0 \text{ for static} \\ \text{vortex}}} (\Phi - a_i \partial_i \Phi)$$

Therefore, the Gauss constraint will become

$$\dot{a}_j \partial_j \partial_i A_i - ig \dot{a}_i \Phi^* \partial_i \Phi + ig \dot{a}_i \Phi \partial_i \Phi^* = 0$$

$$\dot{a}_j \partial_j \partial_i A_i - ig \dot{a}_i \Phi^* \partial_i \Phi + ig \dot{a}_i \Phi \partial_i \Phi^* = 0$$

- This equation is not satisfied:

$$\partial_i A_i \xrightarrow{r \to \infty} -\frac{n}{g} \cdot \partial_i \varepsilon_{ij} \frac{r^j}{r}$$
$$= -\frac{n}{g} \cdot \varepsilon_{ij} \left(\frac{1}{r} \delta_{ij} - \frac{r^i r^j}{r^3} \right) = 0$$

$$\partial_i \Phi \xrightarrow{r \to \infty} v \cdot \partial_i e^{i\Theta} = i v e^{i\Theta} \partial_i \Theta$$

In Gauss constraint:

$$vg (\partial_i \Theta) \dot{a}_i + vg (\partial_i \Theta) \dot{a}_i \neq 0$$

(3)

- With this change we have

$$F_{i0} \longmapsto \partial_i A_0 - \partial_t A_i + \dot{a}_j \partial_j A_i + a_j \partial_j \partial_t A_i + \frac{1}{g} \partial_i \dot{\alpha}$$

insert vortex solution

$$= \dot{a}_j \partial_j A_i - \dot{a}_j \partial_i A_j = \dot{a}_j F_{ji}$$

$$D_0 \Phi \longmapsto \partial_t \Phi - \dot{a}_i \partial_i \Phi + ig A_0 (\Phi - a_i \partial_i \Phi + i \alpha \Phi) + i (\partial_t \alpha) \Phi$$

insert vortex solution

$$= - \dot{a}_i \partial_i \Phi - ig \dot{a}_i A_i \Phi = - \dot{a}_i D_i \Phi$$

- The Gauss constraint becomes

$$\dot{a}_j \partial_i F_{ji} - ig \Phi^* D_i \Phi \dot{a}_i + ig \Phi (D_i \Phi)^* \dot{a}_i = 0$$

This is satisfied by equation (1) and thus the Gauss constraint is invariant under this transformation.

(4)

- We found

$$\Phi(x) \longmapsto \Phi(x) - a_i(t)\, D_i \Phi(x)$$

$$A_i(x) \longmapsto A_i(x) + F_{ij}\, a_j$$

$$\to \partial_t \Phi(x) = \lim_{\Delta t \to 0} \frac{\Phi(x_i - a_i(\Delta t)) - \Phi(x_i)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{-a_i(\Delta t)\, D_i \Phi(x)}{\Delta t} = -\dot{a}_i\, D_i \Phi$$

Similarly, $\partial_t A_i(x) = \dot{a}_j\, F_{ij}$

- $T = \int d^2x \left(\frac{1}{2} (\partial_t A_i)^2 + |\partial_t \Phi|^2 \right)$

$$= \int d^2x \left(\frac{1}{2} F_{ki} F_{kj} + (D_i \Phi)^* (D_j \Phi) \right) \dot{a}_i \dot{a}_j$$

(5)

- $(D_i \Phi)^* (D_j \Phi)\, \dot{a}_i \dot{a}_j$

$$= (D_1 \Phi)^* (D_1 \Phi)\, \dot{a}_1 \dot{a}_1 + (D_1 \Phi)^* (D_2 \Phi)\, \dot{a}_1 \dot{a}_2$$

$$+ (D_2 \Phi)^* (D_1 \Phi)\, \dot{a}_2 \dot{a}_1 + (D_2 \Phi)^* (D_2 \Phi)\, \dot{a}_2 \dot{a}_2 \quad \text{with} \quad D_1 \Phi = -i D_2 \Phi$$

$$= (D_1 \Phi)^* (D_1 \Phi)\, \dot{a}_1 \dot{a}_1 + i\, \cancel{(D_2 \Phi)^* (D_2 \Phi)}\, \dot{a}_1 \dot{a}_2$$

$$- (D_2 \Phi)^* \cancel{(D_2 \Phi)}\, \dot{a}_2 \dot{a}_1 + (D_2 \Phi)^* (D_2 \Phi)\, \dot{a}_2 \dot{a}_2$$

$$= \frac{1}{2} |D_k \Phi|^2\, \delta_{ij}\, \dot{a}_i \dot{a}_j$$

- $F_{ik} F_{jk}\, \dot{a}_i \dot{a}_j$

$$= F_{1k} F_{1k}\, \dot{a}_1 \dot{a}_1 + \overbrace{F_{1k} F_{2k}}^{=0}\, \dot{a}_1 \dot{a}_2$$

$$= \overbrace{F_{1k} F_{1k} \dot{a}_1 \dot{a}_1 + F_{1k} F_{2k}}^{=0} \dot{a}_1 \dot{a}_2$$

$$+ \underbrace{F_{2k} F_{1k}}_{=0,\ \text{because}\ F_{21} = 0 = F_{12}} \dot{a}_2 \dot{a}_1 + F_{2k} F_{2k} \dot{a}_2 \dot{a}_2$$

$$= \frac{1}{2} F_{mn} F_{mn} \delta_{ij} \dot{a}_i \dot{a}_j$$

$$\rightarrow T = \int d^2x \left(\frac{1}{4} F_{mn} F_{mn} + \frac{1}{2} |D_k \Phi|^2 \right) \delta_{ij} \dot{a}_i \dot{a}_j$$

with $F_{mn} F_{mn} = 2B^2 = 4V$ (↑ BPS equation)

$$= \int d^2x \, \frac{1}{2} \left(\frac{1}{4} F_{mn} F_{mn} + |D_k \Phi|^2 + V \right)$$

$$= \frac{1}{2} \cdot M_{vortex} \, \dot{a}_i^2$$

$$= \frac{1}{2} M_{vortex} \, g_{ij} \dot{a}_i \dot{a}_j$$

with $g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

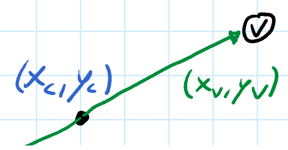
(6)

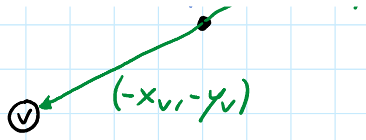
- The geometric form of the moduli space is flat and thus a single vortex moves on straight lines.

Problem 3:

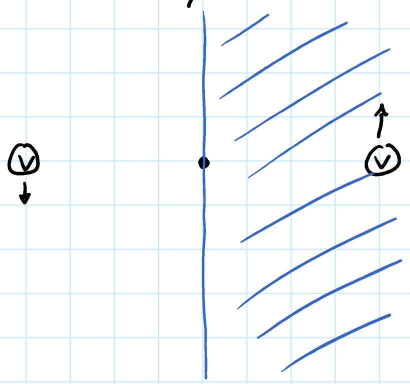
(1)

- $\mathbb{C}$: two coordinates that describe the center of mass
- $\mathbb{C}/\mathbb{Z}_2$: two coordinates for the position of the vortices with respect to the center of mass.



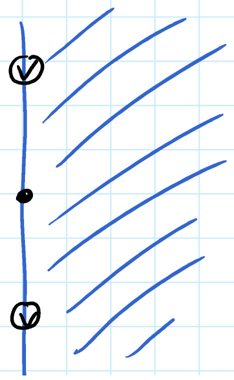


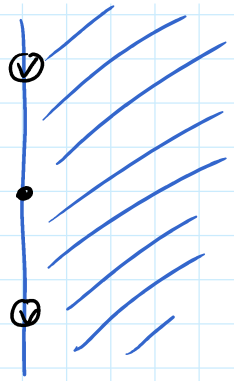
- We have the $\mathbb{Z}_2$, because half of the coordinates always determine the whole system:



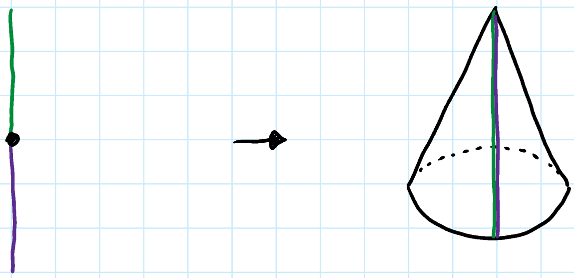
If I move the left vortex down, the right vortex moves up. If I move the left vortex to the right-hand side, then this is the same as saying that the same vortex appears 180° degrees rotated in the left-hand side, because the two vortices are identical.

- In total I can say that the coordinates corresponding to $\mathbb{C}/\mathbb{Z}_2$ describe two vortices at the same time:



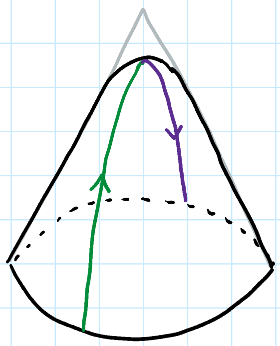


We can say that the two vortices have the same position in the moduli space. Therefore, we can 'glue' the lines together



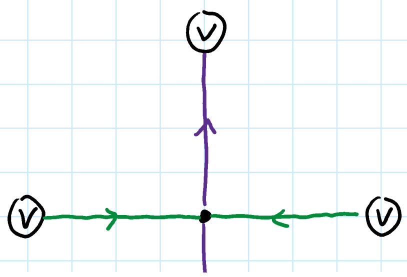
→ $\mathbb{C}/\mathbb{Z}_2$ has the form of a cone

(2)

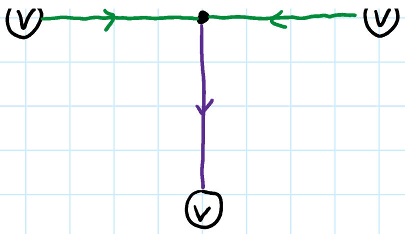


A head-on collision means that the vortices will reach the top of the cone. Since the direction cannot just change, the geodesic goes over the cone on the other side back down.

(3)

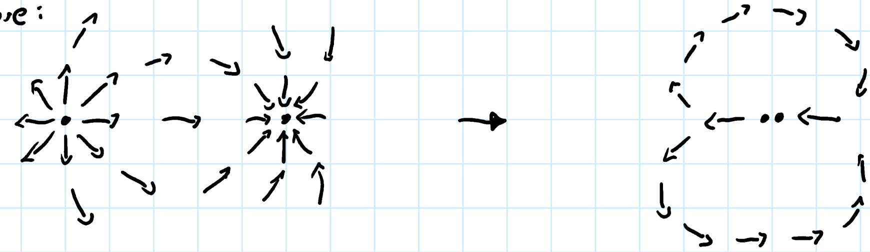


The two vortices collide by 90 degrees.

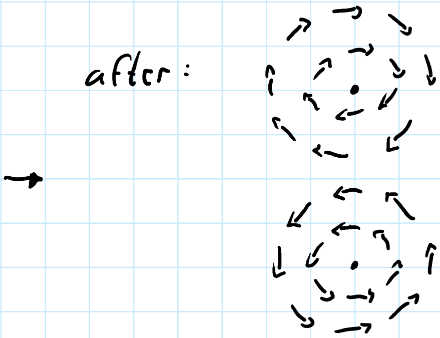


(4)

before:



after:

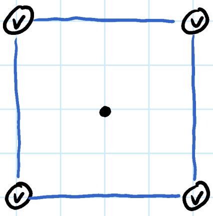


(5)

- 4 vortices in 2 spatial dimensions

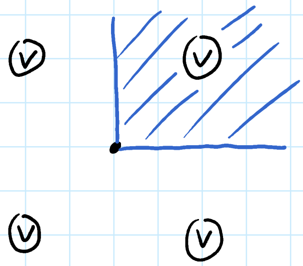
$\rightarrow$ 8 collective coordinates

(6)



- We have the two center of mass coordinates and if we assume that the two vortices are always on a square, two further coordinates are enough to describe the position of all are enough to describe the position of all vortices

→ This subspace has 4 collective coordinates

(7) 

this space is enough to describe all vortices

$$\rightarrow \mathcal{M}_{subspace} = \mathbb{C}_2 \times \frac{\mathbb{C}_2}{\mathbb{Z}_4}$$

- We have again a cone, but sharper than before. The vortices will scatter by 45°.