Topological Defects

Problem Sheet 4 13 November 2023

1. Gravitational Potential of the Domain Wall

In this problem, we will show how gravitation acts on a point mass that is in a system with a domain wall. Note that in this problem we will work in the mostly minus convention for the metric. Furthermore, we will work in 3+1 dimensions.

1. *(optional)* Starting from the Einstein equation, show that in the Newtonian limit, the potential satisfies the equation

$$\vec{\nabla}^2 V = 8\pi G \left(T_{00} - \frac{1}{2} T \right), \tag{1}$$

where T is the trace of the energy-momentum tensor $T_{\mu\nu}$.

2. The Hilbert energy-momentum tensor is defined by

$$T_{\mu\nu} = 2 \frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} - g_{\mu\nu} \mathcal{L}. \tag{2}$$

Calculate the energy-momentum tensor for a domain wall that is located in the $y - z$ plane at $x = 0$. The theory that you should consider is given by the following Lagrangian density

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi)^2 - \frac{\lambda}{4} (\phi^2 - v^2)^2. \tag{3}$$

3. Is the gravitational field of the wall attractive or repulsive ?
4. By solving equation (1) show that the potential is linearly growing for large $|m_h x|$.
Hint : You can use Mathematica for solving the integrals.

2. Derrick's Theorem

First, let us consider n scalar fields ϕ^a with $a = 1, \ldots, n$ in $d + 1$ dimensions with the Lagrangian

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi^a \partial^\mu \phi^a - V(\phi^a). \tag{4}$$

As we already know from domain walls, there may be classical static solutions (solitons) ϕ^a_{cl} that minimize the energy $E = T + U$ with $T = \int \mathrm{d}^d x \frac{1}{2} \partial_i \phi^a \partial_i \phi^a$ and $U = \int \mathrm{d}^d x V(\phi^a)$.

1. Now take $\phi^a_\lambda(x) = \phi^a_{cl}(\lambda x)$ with λ a positive real number. Show that the energy is $E(\lambda) = \lambda^{2-d} T + \lambda^{-d} U$.

2. A soliton extremizes the energy. What does this imply for λ ?
3. Explain why for $d > 1$ there are no solitons allowed.

Consider now a gauge theory in $d+1$ dimensions with Lagrangian

$$\mathcal{L} = -\frac{1}{2}\,\mathrm{Tr}\,(G_{\mu\nu}G^{\mu\nu}) + \mathrm{Tr}\,\big((D_\mu\phi)^\dagger(D^\mu\phi)\big) - V(\phi), \tag{5}$$

with $G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$ and $D_\mu\phi = \partial_\mu\phi + g[A_\mu, \phi]$. The energy is given by $E = G + T + U$ with $G = \int \mathrm{d}^d x \frac{1}{2}\,\mathrm{Tr}(G_{ij}G_{ij})$, $T = \int \mathrm{d}^d x\,\mathrm{Tr}\big((D_i\phi)^\dagger(D_i\phi)\big)$ and $U = \int \mathrm{d}^d x V(\phi)$.

Comment : The theory here is a non-Abelian gauge theory. We will explore the basics on a later problem sheet. The current problem you can solve without knowing these basics.

4. Take the scale transformation $\phi_\lambda(x) = \phi_{cl}(\lambda x)$ and $A^i_\lambda(x) = \lambda A^i_{cl}(\lambda x)$ and show that the energy is $E(\lambda) = \lambda^{4-d}G + \lambda^{2-d}T + \lambda^{-d}U$.
5. For which dimensions you can have solitons ?

3. The Nielsen-Olesen Vortex

When we talk about vortices, we consider a $U(1)$ gauge theory with a complex scalar field ϕ in 2+1 dimensions. The Lagrangian is given by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^*(D^\mu\phi) - \lambda(|\phi|^2 - v^2)^2, \tag{6}$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $D_\mu\phi = \partial_\mu\phi + igA_\mu\phi$.

1. What are the gauge boson mass and Higgs bosons after symmetry breaking ?
2. Write down an expression for the energy corresponding to a static configuration. Which conditions $|\phi|$ and $D_i\phi$ have to satisfy ?
3. A non-trivial solution is given by $\phi \xrightarrow{r\to\infty} ve^{in\theta}$, where (r,θ) are polar coordinates. Find the solution for $A_i(r\to\infty)$?
4. Show that this vortex solution contains a magnetic flux $\Phi = \frac{2\pi n}{g}$.
5. Draw the scalar field ϕ as a vector $(\mathrm{Re}\,\hat{\phi}, \mathrm{Im}\,\hat{\phi})^T$ in a vector plot for $n = 1$, $n = -1$, and $n = 2$.
6. Can you explain qualitatively the Kibble mechanism for vortices ?
7. Is there still a topological defect solution if we extend the theory to 3+1 dimensions ?