Topological Defects

Problem Sheet 3

6 November 2023

1. Domain Walls in ϕ^6 Theory

The ϕ^6 theory is given by the following Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \lambda \phi^2 (\phi^2 - v^2)^2. \tag{1}$$

1. Find the domain wall solutions $(0, \pm v)$ and $(\pm v, 0)$ that correspond to solutions that interpolate between 0 and $\pm v$, and $\pm v$ and 0 respectively.
2. What is the energy of the domain walls ?

2. Interactions between DWs in ϕ^6 Theory

Take the same theory as in the previous problem. You found domain walls that separate regions with vacuum expectation value zero from regions with vacuum expectation value v. In other words, they are the border between symmetric phases and Higgsed phases. In this problem, it is sometimes useful to use Mathematica.

1. Out of the domain wall solutions construct configurations, consisting of two domain walls separated by a distance h, of the following form :
   (a) $(0, v, 0)$
   (b) $(v, 0, v)$
   (c) $(v, 0, -v)$

   Note that these configurations are not static, since they don't solve the static field equations.
2. One way to build a $(v, 0, v)$ configuration is

$$\phi_{(v,0,v)}(x) = \frac{1}{v}\phi_{(0,v)}(x)\phi_{(-v,0)}(x-h) + v. \tag{2}$$

   Why this ansatz is not a good choice ?
3. By considering the total energy of the three cases given in 1 for different distances h, try to predict what happens to the walls if you let them evolve in time.
4. Take the case (b) and show that the interaction between the two walls is exponentially suppressed with the distance h.

3. Coleman-Weinberg Potential

Consider a $U(1)$ gauge theory with the following potential for the scalar field

$$V(\phi) = m^2(T)|\phi|^2 + \frac{m_0^2}{\nu\sigma^2}|\phi|^4 \ln\frac{|\phi|^2}{\sigma^2}, \tag{3}$$

with $m^2(T) = m_0^2 + \frac{1}{4}g^2T^2$. Here g is the gauge coupling, σ a parameter with the dimension of mass (the renormalization scale), and $\nu = \frac{16\pi^2 m_0^2}{3g^4\sigma^2}$.
The mass term is temperature-dependent due to thermal corrections. The second term in the potential is generated radiatively (at one loop level) due to the interaction of the scalar field with the gauge boson. How does the shape of the potential change with decreasing temperature ? Find the temperature T_1 at which minima beside $\phi = 0$ appear. Find the temperature T_C at which these minima are energetically lower than the minimum at $\phi = 0$. Compare the potential at $T = T_C$ with the potential given in Problem 1.

4. Bubble Nucleation

Take the ϕ^6 theory of problem 1. But now we will add a second spatial dimension. With the domain wall solution that you already found, we can build a vacuum bubble of radius R :

$$\phi_{\text{bubble}} = \phi_{(\pm v,0)}(r - R), \tag{4}$$

where inside the bubble the symmetry is broken and outside the symmetry is restored.

1. The bubble will collapse. Use simple relativistic calculations in the thin wall approximation to find $R(t)$.
   Hint : Note that the energy of the domain wall that you calculated before is now an energy density σ_{DW} (also called tension).
2. What term can you add to the potential to obtain a growing bubble ?
3. As we have seen it is possible to create bubbles that grow. These bubbles can arise in the early universe through thermal fluctuations. With this new knowledge, can you describe what happens to a system with the Coleman-Weinberg potential in the different temperature periods ?
4. Looking ahead, do you have any thoughts on how we might be able to observe these so-called first-order phase transitions experimentally in the distant future ?