## Problem 1:

- $\mathcal{L} = (\partial_\mu \Phi)^* (\partial^\mu \Phi) - m^2 |\Phi|^2 + c|\Phi|^4$

  $U(1)$ global symmetry: $\Phi \longmapsto e^{i\Theta}\Phi$

  $2+1$ dimensions

(1)

- $[S] = \overset{M^1}{[\text{energy}]} \cdot \overset{M^{-1}}{[\text{time}]} = M^0$ (you can also use the notation $[S]=0$)

  $S = \int d^{d+1}x\, \mathcal{L} \rightarrow [\mathcal{L}] = M^{d+1}$

  $[\partial_\mu] = M^1 \rightarrow [\Phi] = M^{(d-1)/2}$

- $[c|\Phi|^4] \overset{!}{=} M^{d+1} \rightarrow [c] = M^{3-d}$

(2)

- Euler-Lagrange equation:

$$\partial_\mu \left( \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^*)} \right) - \frac{\partial \mathcal{L}}{\partial \Phi^*} = 0$$

$$\partial_\mu \partial^\mu \Phi + m^2 \Phi - 2c|\Phi|^2 \Phi = 0$$

$$(\Box + m^2 - 2c|\Phi|^2)\Phi = 0$$

- With extremisation of action:

  $\Phi \longmapsto \Phi + \delta\Phi$

$$\delta S = S[\Phi + \delta\Phi] - S[\Phi]$$

$$\int d^4x \Big( \partial_\mu \Phi^* \partial^\mu \delta\Phi + \partial_\mu \delta\Phi^* \partial^\mu \Phi - m^2 \Phi^* \delta\Phi - m^2 \delta\Phi^* \Phi$$

$$+ 2c|\Phi|^2 \Phi^* \delta\Phi + 2c|\Phi|^2 \delta\Phi^* \Phi + \mathcal{O}(\delta\Phi^2) \Big)$$

$$= \int d^4x \Big( (-\Box\Phi^* - m^2 \Phi^* + 2c|\Phi|^2 \Phi^*) \delta\Phi$$

$$= \int d^4x \, \Big( \big(-\Box\Phi^* - m^2\Phi^* + 2c|\Phi|^2\Phi^*\big)\delta\Phi$$
$$+ \big(-\Box\Phi - m^2\Phi + 2c|\Phi|^2\Phi\big)\delta\Phi^* \Big) \overset{!}{=} 0$$

$$\rightarrow \quad \Box\Phi + m^2\Phi - 2c|\Phi|^2\Phi = 0$$

(3)

- $$\mathcal{J}^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\delta\Phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi^*)}\delta\Phi^*$$

- $$\delta\Phi = e^{i\theta}\Phi - \Phi \approx (1+i\theta)\Phi - \Phi = i\theta\Phi$$

$$\rightarrow \quad \mathcal{J}^\mu \sim \partial^\mu\Phi^* \cdot i\theta\Phi + \partial^\mu\Phi\,(-i\theta\Phi^*)$$

$\theta$ arbitrary: $\quad \theta = 1$

$$\mathcal{J}^\mu = i\Phi^*\partial^\mu\Phi - i\Phi\partial^\mu\Phi^*$$

- $$\partial_\mu\mathcal{J}^\mu = i\,\cancel{\partial_\mu\Phi^*\partial^\mu\Phi} + i\Phi^*\Box\Phi - i\,\cancel{\partial_\mu\Phi\partial^\mu\Phi^*} - i\Phi\Box\Phi^*$$

$$\overset{\text{e.o.m.}}{=} i\Phi^*\big(-m^2\Phi + 2c|\Phi|^2\Phi\big) - i\Phi\big(-m^2\Phi^* + 2c|\Phi|^2\Phi^*\big)$$

$$= 0$$

$\rightarrow$ $\mathcal{J}^\mu$ is conserved

(4)

- We add a gauge field $A_\mu$:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\Phi)^*(D^\mu\Phi) - m^2|\Phi|^2 + c|\Phi|^4$$

gauge symmetry:

$$\Phi \longmapsto e^{i\theta(x)}\,, \qquad A_\mu \longmapsto A_\mu - \frac{1}{g}\partial_\mu\theta(x)$$

- covariant derivative transforms in the same way as $\Phi$:

- covariant derivative transforms in the same way as $\Phi$:

$$D_\mu \Phi \longmapsto e^{i\Theta(x)} D_\mu \Phi$$

$$\partial_\mu \Phi \longmapsto \partial_\mu \left(e^{i\Theta(x)} \Phi\right)$$
$$= e^{i\Theta(x)} \partial_\mu \Phi + e^{i\Theta(x)} \Phi \, i \left(\partial_\mu \Theta(x)\right)$$

We can compensate the new term with

$$D_\mu \Phi = \partial_\mu \Phi + ig A_\mu \Phi$$

- The Lagrangian is now gauge invariant:

$$F_{\mu\nu} \longmapsto \partial_\mu A_\nu - \frac{1}{g} \cancel{\partial_\mu \partial_\nu} \Theta - \partial_\nu A_\mu + \frac{1}{g} \cancel{\partial_\nu \partial_\mu} \Theta$$
$$= \partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu}$$

$$(D_\mu \Phi)^* (D^\mu \Phi) \longmapsto (D_\mu \Phi)^* e^{-i\Theta} \cdot e^{+i\Theta} (D_\mu \Phi) = (D_\mu \Phi)^* (D^\mu \Phi)$$

$$|\Phi|^2 \longmapsto \Phi^* e^{-i\Theta} \cdot e^{+i\Theta} \Phi = |\Phi|^2$$

$$\Rightarrow \mathcal{L} \longmapsto \mathcal{L}$$

(5)

- $$[D_\mu, D_\nu] \Phi = (\partial_\mu + ig A_\mu)(\partial_\nu + ig A_\nu)\Phi - (\partial_\nu + ig A_\nu)(\partial_\mu + ig A_\mu)\Phi$$
$$= \partial_\mu \cancel{\partial_\nu \Phi} + ig(\partial_\mu A_\nu)\Phi + ig \cancel{A_\nu} \partial_\mu \Phi$$
$$+ ig \cancel{A_\mu} \partial_\nu \Phi - g^2 \cancel{A_\mu} A_\nu \Phi$$
$$- \partial_\nu \cancel{\partial_\mu} \Phi - ig(\partial_\nu A_\mu)\Phi - ig \cancel{A_\mu} \partial_\nu \Phi$$
$$- ig A_\nu \cancel{\partial_\mu} \Phi + g^2 \cancel{A_\nu} A_\mu \Phi$$

$$= ig\Phi (\partial_\mu A_\nu - \partial_\nu A_\mu) = ig \Phi F_{\mu\nu}$$

(6)

- $\partial_\mu \left( \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} \right) - \frac{\partial \mathcal{L}}{\partial \phi^*} = 0$

$$\partial_\mu (D^\mu \phi) + ig A_\mu (D^\mu \phi) + m^2 \phi - 2c |\phi|^2 \phi = 0$$

$$(D_\mu D^\mu + m^2 - 2c |\phi|^2) \phi = 0$$

- $\partial_\mu \left( \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} \right) - \frac{\partial \mathcal{L}}{\partial A_\nu} = 0$

$$\partial_\mu \left( -\frac{1}{4} \cdot 2 F^{\alpha\beta} \cdot \frac{\partial}{\partial(\partial_\mu A_\nu)} \left( 2 \partial_\alpha A_\beta \right) \right) - \left( -ig \phi^* D^\nu \phi + ig \phi D^\nu \phi^* \right) = 0$$

$$-\partial_\mu F^{\mu\nu} + ig \phi^* D^\nu \phi - ig \phi (D^\nu \phi)^* = 0$$

(7)

Noether's theorem:

If a Lagrangian is invariant under a specific up to a total divergence, then there is a conserved current given by

$$J^\mu = \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \delta \phi - K^\mu$$

- derivation energy momentum tensor:

energy momentum tensor corresponds to invariance under time and space transformations $x^\mu \longmapsto x^\mu + a^\mu$

Let $a^\mu$ be infinitesimally small

$$\varphi^i(x+a) \approx \varphi^i(x) + \underbrace{a^\mu \partial_\mu \varphi^i(x)}_{}$$

$$\varphi^i(x+a) \approx \varphi^i(x) + \underbrace{a^\mu \partial_\mu \varphi^i(x)}_{=\delta\varphi^i}$$

And $$\mathcal{L}(x+a) \approx \mathcal{L}(x) + a^\mu \partial_\mu \mathcal{L}(x)$$
$$= \mathcal{L}(x) + \partial_\mu \big(\underbrace{a^\mu \mathcal{L}(x)}_{K^\mu}\big)$$

Noethers theorem gives conserved current
$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi^i)} a^\nu \partial_\nu \varphi^i - a^\mu \mathcal{L}$$

$$\rightarrow T^\mu{}_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi^i)} \partial_\nu \varphi^i - \delta^\mu{}_\nu \mathcal{L}$$

- Applying this to the given Lagrangian:

$$T^\mu{}_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\gamma)} \partial_\nu A_\gamma + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \partial_\nu \Phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^*)} \partial_\nu \Phi^* - \delta^\mu{}_\nu \mathcal{L}$$
$$= -F^{\alpha\beta} \eta^\mu{}_\alpha \eta^\gamma{}_\beta \partial_\nu A_\gamma + (D^\mu \Phi)^* \partial_\nu \Phi + (D^\mu \Phi) \partial_\nu \Phi^*$$
$$+ \delta^\mu{}_\nu \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} - \delta^\mu{}_\nu (D_\alpha \Phi)^* (D^\alpha \Phi)$$
$$+ \delta^\mu{}_\nu (m^2 |\Phi|^2 - c|\Phi|^4)$$

- The first line is not gauge invariant.

$$-F^{\mu\alpha} \partial_\nu A_\alpha + (D^\mu \Phi)^* \partial_\nu \Phi + (D^\mu \Phi) \partial_\nu \Phi^*$$

A gauge invariant expression would be

$$-F^{\mu\alpha} F_{\nu\alpha} + (D^\mu \Phi)^* (D_\nu \Phi) + (D^\mu \Phi)(D_\nu \Phi)^*$$
$$= -F^{\mu\alpha} \partial_\nu A_\alpha + (D^\mu \Phi)^* \partial_\nu \Phi + (D^\mu \Phi) \partial_\nu \Phi^*$$
$$+ F^{\mu\alpha} \partial_\alpha A_\nu + (D^\mu \Phi)^* i g A_\nu \Phi - (D^\mu \Phi) i g A_\nu \Phi^*$$

The green part is conserved:

$$\partial_\mu (\ldots) = \underline{\partial_\mu F^{\mu\alpha}} \partial_\alpha A_\nu + F^{\mu\alpha} \cancel{\partial_\mu \partial_\alpha} A_\nu$$

$$+ \partial_\mu (D^\mu\Phi)^* ig A_\nu \Phi + (D^\mu\Phi)^* ig (\partial_\mu A_\nu) \Phi + (D^\mu\Phi)^* ig A_\nu \partial_\mu \Phi$$

$$- \partial_\mu (D^\mu\Phi) ig A_\nu \Phi^* - (D^\mu\Phi) ig (\partial_\mu A_\nu) \Phi^* - (D^\mu\Phi) ig A_\nu \partial_\mu \Phi^*$$

e.o.m. for $A_\mu$

$$= \underline{(ig \Phi^* \cancel{D^\mu}\Phi - ig \Phi \cancel{D^\mu} \Phi^*)} \partial_\mu A_\nu$$

$$+ (D^\mu\Phi)^* \cancel{ig (\partial_\mu A_\nu) \Phi} - ig (D^\mu\Phi) \cancel{ig} (\partial_\mu A_\nu) \Phi^*$$

$$+ \partial_\mu (D^\mu\Phi)^* ig A_\nu \Phi + (D^\mu\Phi)^* ig A_\nu \partial_\mu \Phi$$

$$- \partial_\mu (D^\mu\Phi) ig A_\nu \Phi^* - (D^\mu\Phi) ig A_\nu \partial_\mu \Phi^*$$

e.o.m. for $\Phi, \Phi^*$

$$= ig A_\mu (D^\mu\Phi)^* ig A_\nu \Phi + (D^\mu\Phi)^* ig A_\nu \partial_\mu \Phi$$

$$+ ig A_\mu (D^\mu\Phi) ig A_\nu \Phi^* - (D^\mu\Phi) ig A_\nu \partial_\mu \Phi^*$$

$$= ig A_\nu \left( (D_\mu\Phi)(D^\mu\Phi)^* - (D_\mu\Phi)^* (D^\mu\Phi) \right) = 0$$

The added part is conserved and thus we can add it to the energy momentum tensor.

$$\Rightarrow T^\mu{}_\nu = - F^{\mu\alpha} F_{\nu\alpha} + (D^\mu\Phi)^* (D_\nu\Phi) + (D^\mu\Phi)(D_\nu\Phi^*)$$

$$+ \delta^\mu{}_\nu \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} - \delta^\mu{}_\nu (D_\alpha\Phi)^* (D^\alpha\Phi)$$

$$+ \delta^\mu{}_\nu (m^2 |\Phi|^2 - c |\Phi|^4)$$

Comment: The Hilbert energy momentum tensor

$$T_{\mu\nu} = +2 \frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} - g_{\mu\nu} \mathcal{L}$$

would give the same result.

(8)

- electric field $E_k = F_{0k}$

  magnetic field $B_k = -\frac{1}{2} \varepsilon_{kij} F_{ij}$

$$\varepsilon_{mnk} B_k = -\frac{1}{2} \varepsilon_{mnk} \varepsilon_{kij} F_{ij}$$

$$-2 \varepsilon_{mnk} B_k = F_{mn} - F_{nm}$$

$$\rightarrow F_{ij} = -\varepsilon_{ijk} B_k$$

- energy density $\varepsilon = T^0{}_0$

$$\varepsilon \supset -F^{0\alpha} F_{0\alpha} + \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta}$$

$$= F_{0i} F_{0i} - \frac{1}{2} F_{0i} F_{0i} + \frac{1}{4} F_{ij} F_{ij}$$

$$= \frac{1}{2} E_i^2 + \frac{1}{4} \varepsilon_{ijk} B_k \varepsilon_{ijm} B_m$$

$$= \frac{1}{2} (E_i^2 + B_i^2)$$

## Problem 2:

(1)

- $\mathcal{L} \supset + 2\lambda v^2 |\Phi|^2 \equiv m_\Phi^2 |\Phi|^2$

  $\rightarrow$ mass of field $\Phi$ is $m_\Phi = \sqrt{2\lambda}\, v$

  $\Phi = \Phi_{real} + i\Phi_{imag}$ $\rightarrow$ 2 degrees of freedom

- there is no mass term for $A_\mu$

  $\rightarrow$ massless gauge field

  $\rightarrow$ 2 degrees of freedom

(2)

- $D_\mu \Phi = \partial_\mu \left( (N h(x) + v) e^{i\Theta(x)} \right) + i g A_\mu (N h(x) + v) e^{i\Theta(x)}$

• $D_\mu \Phi = \partial_\mu \left( (N h(x) + v) e^{i\Theta} \right) + ig A_\mu (N h(x) + v) e^{i\Theta}$

$$= N \partial_\mu h \, e^{i\Theta} + (N h + v) e^{i\Theta} \, ig \left(A_\mu + \frac{1}{g} \partial_\mu \Theta\right)$$

• $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + N^2 \partial_\mu h \partial^\mu h + (N h + v)^2 g^2 \left(A_\mu + \frac{1}{g} \partial_\mu \Theta\right)^2$
$$- \lambda \left( (N h + v)^2 - v^2 \right)^2$$

canonical normalization $\rightarrow$ $N = \frac{1}{\sqrt{2}}$

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \partial_\mu h \partial^\mu h + \left(\frac{h}{\sqrt{2}} + v\right)^2 g^2 \left(A_\mu + \frac{1}{g} \partial_\mu \Theta\right)^2$$
$$- \lambda \left( \frac{1}{2} h^2 + \sqrt{2} h v \right)^2$$

(3)

• $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ with $A_\mu = B_\mu - \frac{1}{g} \partial_\mu \Theta$
$$= \partial_\mu B_\nu - \frac{1}{g} \partial_\mu \partial_\nu \Theta - \partial_\nu B_\mu + \frac{1}{g} \partial_\nu \partial_\mu \Theta$$
$$= \partial_\mu B_\nu - \partial_\nu B_\mu$$

$\rightarrow$ $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \partial_\mu h \partial^\mu h + \left(\frac{h}{\sqrt{2}} + v\right)^2 g^2 B_\mu B^\mu$
$$- \lambda \left( \frac{1}{2} h^2 + \sqrt{2} h v \right)^2$$

where $F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$

(4)

• $\mathcal{L} \supset v^2 g^2 B_\mu B^\mu - 2 \lambda v^2 h^2$
$$\equiv \frac{1}{2} m_V^2 B_\mu B^\mu - \frac{1}{2} m_h^2 h^2$$

$\rightarrow$ $m_h = 2\sqrt{\lambda} v$

$m_V = \sqrt{2} g v$

• massive vector field $B_\mu$ $\rightarrow$ 3 degrees of freedom

real scalar field $h$ $\rightarrow$ 1 degree of freedom real scalar field $h$ $\to$ 1 degree of freedom

$\Rightarrow$ number of degrees of freedom did not change

- In this case the gauge boson obtained one degree of freedom coming from the scalar field.
  If the theory hadn't have the gauge field, we would have obtained the massive Higgs field $h$ and a massless Goldstone boson $\Theta$.