Topological Defects

Problem Sheet 1 23 October 2023

1. Complex Scalar Theory

Let us start with a complex scalar field theory with $U(1)$ global symmetry in $d+1$ dimensions. The Lagrangian is given by

$$\mathcal{L} = (\partial_\mu \phi)^*(\partial^\mu \phi) - m^2|\phi|^2 + c|\phi|^4. \tag{1}$$

1. What is the mass unit of the constant c?
2. Derive the field equations for ϕ.
3. Calculate the Noether current and show that it is conserved.

Instead of the global $U(1)$ symmetry $\phi \mapsto e^{i\theta}\phi$, we want to have a local symmetry $\phi \mapsto e^{i\theta(x)}\phi$. However, in its current form, the Lagrangian (1) is not invariant under this local transformation. In order to restore this symmetry, we must introduce an additional field into the theory, namely the vector field A_μ. The new Lagrangian is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu \phi)^*(D^\mu \phi) - m^2|\phi|^2 + c|\phi|^4, \tag{2}$$

where D_μ is called the covariant derivative and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength tensor. The vector field transforms under local $U(1)$ transformation (gauge transformation) as $A_\mu \mapsto A_\mu - \frac{1}{g}\partial_\mu \theta(x)$.

4. The covariant derivative of a field ϕ transforms under gauge transformations in the same way as the field ϕ itself. Find the expression for $D_\mu \phi$ and show that the Lagrangian (2) is gauge invariant.
5. Show the following relationship between the field strength tensor and the covariant derivative :

$$[D_\mu, D_\nu] = igF_{\mu\nu}. \tag{3}$$

6. Derive the equations of motion for A_μ and ϕ.
7. Find the gauge invariant energy-momentum tensor of the theory.
8. Take now $d = 3$ and write down the field strength part of the energy density in terms of the electric and magnetic fields, defined by

$$E_k = F_{0k}, \tag{4}$$

$$B_k = -\frac{1}{2}\varepsilon_{kij}F_{ij}. \tag{5}$$

2. Symmetry Breaking in $U(1)$ Gauge Theory

Consider the Lagrangian with $U(1)$ gauge symmetry

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu \phi)^*(D^\mu \phi) - \lambda(|\phi|^2 - v^2)^2. \tag{6}$$

1. Write down the mass of the fields ϕ and A_μ. What are the number of degrees of freedom of these fields ?
2. Insert $\phi(x) = (Nh(x) + v)e^{i\theta(x)}$ into the Lagrangian and determine N such that the Lagrangian is canonically normalized.
3. Let us define the new field $B_\mu = A_\mu + \frac{1}{g}\partial_\mu\theta$ and write the Lagrangian in terms of h and B_μ.
4. What are the masses and degrees of freedom for the fields B_μ and h ?