LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN

Fakultät für Physik im WiSe 2023/24
TMP - TA4: Condensed Matter Many-Body-Physics and Field Theory II
Dozent: Prof. Dr. Fabian Grusdt
Übungen: Nader Mostaan, Lukas Homeier

https://www2.physik.uni-muenchen.de/lehre/vorlesungen/wise_23_24/TMP-TA4/index.html

Sheet 1:

Hand-out: Monday, Oct. 23, 2023; Solutions: Tuesday, Oct. 31, 2023

Problem 1 Fermi gas with scattering potential

In this problem, we apply the Green's function formalism to the scattering of fully spin-polarized electrons off a central potential. Assume an ideal Fermi gas of non-interacting electrons at $T = 0$ with chemical potential μ in d dimensions. The Fermi gas is subject to a central potential $U(\mathbf{x}) = U(x)$, with $x = |\mathbf{x}|$. In second quantization, the physics is governed by the Hamiltonian

$$\hat{H} = \int d^d x\, \hat{\psi}^\dagger(\mathbf{x}) \Big(-\frac{\hbar^2 \nabla^2}{2m} \Big) \hat{\psi}(\mathbf{x}) + \int d^d x\, \hat{\psi}^\dagger(\mathbf{x}) U(\mathbf{x}) \hat{\psi}(\mathbf{x})\,. \tag{1}$$

In Eq. 1, $\hat{\psi}(\mathbf{x})$ is the electron field operator and the first(second) term stands for the electrons kinetic(potential) energy.

(1.a) Rewrite the Hamiltonian in momentum space in terms of fermionic operators $\hat{c}^\dagger_{\mathbf{k}}$ and $\hat{c}_{\mathbf{k}} = \int d^d x \exp\big(-i\mathbf{k}\cdot\mathbf{x}\big)\hat{\psi}(\mathbf{x})$.

(1.b) Write down the Dyson's equation for the electron's Green's function $G(\mathbf{k}, \mathbf{k}', \omega)$ in terms of the electron's free Green's function $G_0(\mathbf{k}, \omega)$ and the Fourier transform of the potential U. Here, $\mathbf{k}'$ and $\mathbf{k}$ stands for electron's initial and final momentum, respectively. Given the diagrammatic representations below, express the Dyson's equation in diagrammatic form.

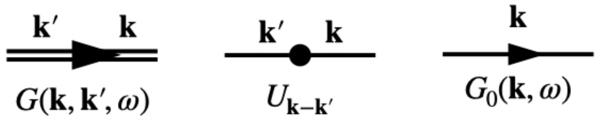


Abbildung 1: Diagrammatic representation of $G(\mathbf{k}, \omega)$, $G_0(\mathbf{k}, \omega)$ and $U_{\mathbf{k}-\mathbf{k}'}$.

(1.c) The geometric summation in Dyson's equation can take an alternative form in terms of the $t - matrix$, represented by

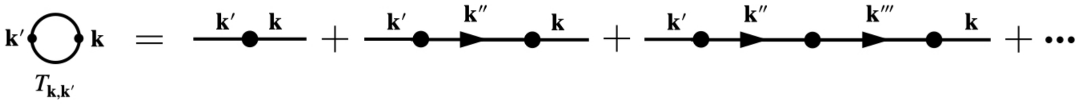


Abbildung 2: Diagrammatic representation of the t-matrix.

Write a self consistent equation satisfied by $T_{\mathbf{k},\mathbf{k}'}(\omega)$.

In the following sections, assume that $U(\mathbf{x})$ is a delta potential, $U(\mathbf{x}) = U\delta^{(d)}(\mathbf{x})$.

(1.d) Argue that for so-called s-wave scattering, $T_{\mathbf{k},\mathbf{k}'}(\omega)$ is independent of $\mathbf{k}$ and $\mathbf{k}'$, $T_{\mathbf{k},\mathbf{k}'}(\omega) = T(\Omega)$.

(1.e) Show that $T(\omega)$ is of the form

$$T(\omega) = \frac{U}{1 - UF(\omega)}\,. \tag{2}$$

Obtain an integral expression for $F(\omega)$ containing $N(\epsilon)$, the density of states at energy ϵ.

(1.f) For $d = 2$, find an exact expression for $F(\omega)$. To avoid divergences in the integral, take a high energy cutoff Λ for the energies involved.

(1.g) Show that for $d = 2$, any attractive potential has a bound state.

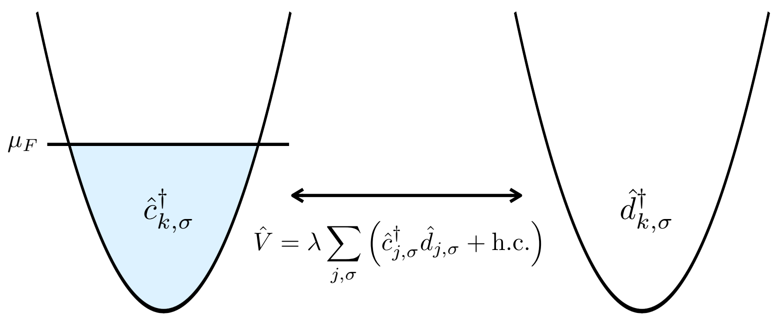
Problem 2 Angle-resolved photoemission spectroscopy (ARPES)

In ARPES the single-particle spectrum can be measured with full energy ω and momentum k resolution. In an experiment, a photon removes an electron from a sample – in this process energy and momentum is conserved.

We model the sample by spin-1/2 fermions $\hat{c}^\dagger_{k,\sigma}$ governed by a generic interacting Hamiltonian $\hat{H}_{\text{sample}}$ at temperature T. The process of removing an electron is described by a weak perturbation $\hat{V}$ that locally annihilates a fermion in the sample and creates a fermion in a (non-interacting) probe system $\hat{d}^\dagger_{j,\sigma}$. The latter is a non-interacting, free fermion system described by

$$\hat{H}_0 = \sum_{k,\sigma} (\epsilon(k) - \mu)\hat{d}^\dagger_{k,\sigma}\hat{d}_{k,\sigma}, \tag{3}$$

with a known dispersion relation $\epsilon(k)$. Here, we consider a 2D square-lattice tight-binding model with $\epsilon(k) = -2t\left[\cos(k_x) + \cos(k_y)\right]$ and an initially empty probe system with $\mu_{\text{probe}} = -4t$. The two systems are initially uncoupled and in equilibrium. At time $t = 0$, we apply the perturbation $\hat{V}$ and calculate the response of the system.



(2.a) Use Fermi's golden rule to calculate the rate $\Gamma(k, \omega)$ under which fermions tunnel from the sample to the bath. What is the valid parameter regime of the result?
Hint: Fermi's golden rule perturbatively describes the rate between an initial and final state given by $\Gamma_{i \to f} = \frac{2\pi}{\hbar} p_i (1 - p_f) |\langle f|\hat{V}|i\rangle|^2 \delta(E_f - E_i)$, where p_n is the probability that the system is in state $|n\rangle$.

(2.b) Find a relation between the rate $\Gamma(k, \omega)$ and the spectral function $A(k, \omega)$. You should find that $\Gamma \propto A$, i.e. the emission rate of ejected electrons which can be measured experimentally, gives direct access to the single-particle spectrum of the sample.

(2.c) Find a relation between the spectral function $A(k, \omega)$ and the Green's function $G(k, \omega)$.
Hint: Use the Lehman representation.

(2.d) Now we assume a sample with non-interacting free fermions $\hat{c}$'s, i.e. $\hat{H}_{\text{sample}} \sim \hat{H}_0$ and we set the temperature to $T = 0$. Sketch the rate $\Gamma(k, \omega)$ for different chemical potentials μ (=filling).

(2.e) Our model describes the conceptual idea of ARPES. Describe in words what else you have to take into account in an actual solid state experiment.