Ludwig-Maximilians-Universität München

Quantum Field Theory (Quantum Electrodynamics)

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Guidelines:

- The exam consists of 6 problems.
- The duration of the exam is 48 hours.
- Please write your name or matriculation number on every sheet that you hand in.
- Your answers should be comprehensible and readable.

GOOD LUCK!

Exercise 1	10 P
Exercise 2	15 P
Exercise 3	30 P
Exercise 4	20 P
Exercise 5	20 P
Exercise 6	20 P

Total	115 P

Problem 1 (10 points)

How many independent real scalar fields enter the following Lagrangians? Justify your answers by explicit computations. Assume φ and χ to be real scalar fields.

Hint: You may find the non-zero eigenvalues of the kinetic and mass matrices.

a)

$$\mathcal{L}_1 = \frac{1}{2}(\partial_\mu \chi)^2 + 2(\partial_\mu \varphi)^2 + 2(\partial_\mu \chi)(\partial^\mu \varphi) \ .$$

b)

$$\mathcal{L}_2 = \mathcal{L}_1 + \frac{1}{2}(\partial_\mu \chi)^2 + \frac{3}{2}(\partial_\mu \varphi)^2 + (\partial_\mu \chi)(\partial^\mu \varphi) - m^2\chi^2 - 2m^2\varphi^2 \ .$$

Problem 2 (15 points)

a) Take the following Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)^2 + a\varphi + b\varphi^2 + c\varphi^3 + d\varphi^4 \ ,$$

with φ a real scalar field and a, b, c, d constants. Assume invariance under the discrete transformation $\varphi \to -\varphi$. What can you say about the parameters a, b, c, d?

b) Take the following Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi^*)(\partial^\mu \varphi) + a(\varphi + \varphi^*) + b(\varphi^2 + \varphi^{*2}) + c(\varphi^3 + \varphi^{*3}) + d(\varphi\varphi^*)^2 \ ,$$

with φ a complex scalar field and a, b, c, d constants. Assume invariance under the discrete transformation $\varphi \to e^{\frac{i2\pi N}{3}}\varphi, N =$ integer. What can you say about the parameters a, b, c, d?

c) The Lagrangian describing fermions interacting with photons is

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} + e_1\bar{\psi}\gamma_\mu\psi A^\mu + e_2\bar{\psi}\gamma_\mu\gamma^5\psi A^\mu \ ,$$

where ψ is a fermion field, $\bar{\psi} = \psi^\dagger\gamma^0$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. If A^μ is a vector under parity and parity is a symmetry of the Lagrangian, what can you say about the coefficients e_1 and e_2?

Problem 3 (30 points)

Consider a theory given by the following Lagrangian in 4 space-time dimensions (we use units $c = \hbar = 1$)

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)(\partial^\mu \varphi) - \frac{1}{2}M^2\varphi^2 + \bar{\psi}(i\partial\!\!\!/ - m)\psi - \frac{\lambda}{2}\varphi^2\bar{\psi}\psi \ ,$$

where φ is a real scalar field of mass M, ψ is a Dirac spinor of mass m with $\bar{\psi} \equiv \psi^\dagger\gamma^0$, and λ is a coupling constant.

a) What are the mass dimensions of the following entities?

   i) ψ
   ii) φ
   iii) λ

b) Find the equations of motion.

c) Check explicitly whether the theory is invariant under the following transformations

   i) $\psi \to \psi' = \mathrm{e}^{-i\alpha}\psi\,,$
   ii) $\psi \to \psi'' = e^{-i\beta\gamma_5}\psi\,,$

   with α, β constants. If yes, derive the corresponding Noether currents and show that they are conserved on the equations of motion.

d) State the Feynman rules for all of the propagators and vertices of this theory. No derivation is necessary.

e) Consider the process $\psi\bar{\psi} \to \varphi\varphi$. Draw and label the Feynman diagram(s) contributing to this process to leading order in λ.

f) Under what conditions is the above process kinematically allowed?

g) Derive the spin-averaged amplitude squared for the above process in terms of the Mandelstam variables.

Bonus (extra 5 pts): Draw and label an example of a tree-level Feynman diagram that is second order in λ describing the process $\psi\bar{\psi} \to \varphi\varphi\varphi\varphi$.

Problem 4 (20 points)

Consider a theory given by the following Lagrangian in 4 space-time dimensions (we use units $c = \hbar = 1$)

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi - e\bar{\psi}\gamma^\mu B_\mu \psi - \frac{1}{4}G_{\mu\nu}G^{\mu\nu} + \frac{\mu^2}{2}B_\mu B^\mu + (D_\mu\varphi)^*(D^\mu\varphi) - M^2\varphi^*\varphi\,,$$

where ψ is a Dirac spinor of mass m with $\bar{\psi} \equiv \psi^\dagger\gamma^0$, B_μ a massive vector field of mass μ, with its field strength-tensor given by $G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$, and φ a massive complex scalar field of mass M. Also, D_μ is defined as $D_\mu = \partial_\mu + igB_\mu$. Finally, e and g are constants.

a) Find the equations of motion.

b) State the Feynman rules for all of the propagators and vertices of the theory. No derivation is necessary.

c) Let ψ transform under a local $U(1)$ symmetry: $\psi \to \psi' = \mathrm{e}^{-i\alpha}\psi$, where $\alpha = \alpha(x)$. Derive explicitly how the other fields have to transform, and what conditions the parameters m, M, μ, e, g have to satisfy, so that the theory be invariant under this transformation.

Hint: Consider first the invariance of $\bar{\psi}(i\partial\!\!\!/ - m)\psi - e\bar{\psi}\gamma^\mu B_\mu \psi$, then the invariance of $(D_\mu\varphi)^*(D^\mu\varphi) - M^2\varphi^*\varphi$, and finally the invariance of the remaining terms.

d) Draw and label an example of a Feynman diagram describing the process $\psi\bar{\psi} \to \varphi\varphi^*$.

Problem 5 (20 points)

a) Show that for the scattering of an electron $e^-(p) \to e^-(p')$ off an external static potential $A_\mu(0, \vec{x})$

$$\langle f|\, S - 1\, |i\rangle = i\mathcal{M}\, 2\pi\delta(E' - E) \equiv ie\, 2\pi\delta(E' - E)\, \bar{u}(p')\gamma^\mu u(p)\tilde{A}_\mu(\vec{q}) \ ,$$

where $q = p' - p$ and $\tilde{A}_\mu(\vec{q}) = \int \mathrm{d}^3x\, \mathrm{e}^{-i\vec{q}\cdot\vec{x}} A_\mu(\vec{x})$ is the Fourier transform of the potential.

Hint: A^μ is an external classical c-number, so it is not involved in any contractions.

b) Next derive the following expression for the cross section (β is the velocity of the incoming electron)

$$\mathrm{d}\sigma = |\mathcal{M}|^2\, 2\pi\delta(E' - E)\frac{1}{2E\beta}\frac{\mathrm{d}^3p'}{(2\pi)^3 2E'} \ .$$

c) With the above equation calculate

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha^2}{4|\vec{p}|^2\beta^2\sin^4\frac{\theta}{2}}\left(1 - \beta^2\sin^2\frac{\theta}{2}\right) \ ,$$

for an electron scattering off the following potential $A_\mu(\vec{x}) = (\frac{e}{4\pi|\vec{x}|}, 0, 0, 0)$. In the above $\alpha = e^2/4\pi$ is the fine-structure constant and θ is the angle between $\vec{p}$ and $\vec{p}'$.

Problem 6 (20 points)

Consider a theory given by the following Lagrangian in 3 space-time dimensions ($c = \hbar = 1$)

$$\mathcal{L} = (\partial_\mu\varphi^*)(\partial^\mu\varphi) - m^2\varphi^*\varphi - \frac{\lambda^2}{2}(\varphi^*\varphi)^2 - \kappa^3(\varphi^*\varphi)^3 \ ,$$

where φ is a complex scalar field, and $m > 0$, $\lambda > 0$, $\kappa > 0$ are constants.

a) What are the mass dimensions of the following entities?

i) φ

ii) m

iii) λ

iv) κ

b) Find the equations of motion.

c) State the Feynman rules for all of the propagators and vertices of this theory. No derivation is necessary.

Consider the process

$$\varphi_{p_1}\varphi^*_{p_2} \to \varphi_{q_1}\varphi^*_{q_2}\varphi_{q_3}\varphi^*_{q_4} \ ,$$

where p_i $(i = 1,2)$, and q_j $(j = 1,2,3,4)$ are the incoming and outgoing momenta, respectively. We define the following variables

$$\begin{aligned} s &= (p_1 + p_2)^2 \ , \\ t_j &= (p_1 + p_2 - q_j)^2 \ , \\ u_j &= (p_1 - q_j - q_{j+1})^2 \ , \end{aligned}$$

for $j = 1,2,3,4$ and $q_5 \equiv q_1$.

d) Show that

$$\sum_{j=1}^{4} t_j - 2s = 4m^2 \, .$$

e) Draw and label a Feynman diagram contributing to this process at tree level.