Ludwig-Maximilians-Universität München

SOLUTIONS TO

Quantum Field Theory (Quantum Electrodynamics)

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Guidelines:

- The exam consists of 7 problems.
- The duration of the exam is 96 hours.
- Please write your name or matriculation number on every sheet that you hand in.
- Your answers should be comprehensible and readable.

GOOD LUCK!

Exercise 1	8 P
Exercise 2	15 P
Exercise 3	25 P
Exercise 4	15 P
Exercise 5	25 P
Exercise 6	15 P
Exercise 7	20 P

Total	123 P

Problem 1 (8 points)

Simplify the following expressions as much as possible without using any representation of the γ matrices. The Minkowski metric convention is $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$. You may use the definitions $P_{\substack{L\\R}} = \frac{1}{2}(1 \mp \gamma^5)$ as well as the identities and relations you encountered during the course.

a) $\text{tr}[\gamma^\mu \gamma_\nu P_L \gamma_\mu P_R \gamma^\nu]$

Solution [2 P]

$$\text{tr}[\gamma^\mu \gamma_\nu P_L \gamma_\mu P_R \gamma^\nu] = \text{tr}[\gamma^\nu \gamma^\mu \gamma_\nu \gamma_\mu P_R] = -2\text{tr}[\gamma^\mu \gamma_\mu P_R] = -8\text{tr}[P_R] = \boxed{-16}$$

b) $\text{tr}[(\gamma^\mu)^\dagger \gamma_\sigma \gamma_\nu P_R \gamma_\rho \gamma^5 P_L \gamma^\nu \gamma_\mu]$

Solution [2 P]

$$\begin{aligned}
&\text{tr}[(\gamma^\mu)^\dagger \gamma_\sigma \gamma_\nu P_R \gamma_\rho \gamma^5 P_L \gamma^\nu \gamma_\mu] = \text{tr}[\gamma_\mu \gamma^0 \gamma^\mu \gamma^0 \gamma_\sigma \gamma_\nu \gamma_\rho P_L \gamma^5 P_L \gamma^\nu] = \\
&= \text{tr}[\gamma_\mu (2\eta^{\mu 0} - \gamma^\mu \gamma^0) \gamma^0 \gamma_\sigma \gamma_\nu \gamma_\rho \gamma^5 \gamma^\nu P_R] = -\text{tr}[\gamma_\mu (2\eta^{\mu 0}\gamma^0 - \gamma^\mu) \gamma_\sigma \gamma_\nu \gamma_\rho \gamma^\nu \gamma^5 P_R] = \\
&= -\text{tr}[(2-4)\gamma_\sigma (2\eta_{\nu\rho} - \gamma_\rho \gamma_\nu) \gamma^\nu P_R] = 2\text{tr}[\gamma_\sigma (2\gamma_\rho - 4\gamma_\rho) P_R] = \\
&= -4\text{tr}[\gamma_\sigma \gamma_\rho \frac{1}{2}(1+\gamma^5)] = -2\text{tr}[\gamma_\sigma \gamma_\rho] = \boxed{-8\eta_{\rho\sigma}}
\end{aligned}$$

c) $\text{tr}[\eta_{\mu\nu} \gamma^\rho \gamma_\sigma]$

Solution [2 P]

$$\text{tr}[\eta_{\mu\nu} \gamma^\rho \gamma_\sigma] = \eta_{\mu\nu} \text{tr}[\gamma^\rho \gamma^\alpha \eta_{\alpha\sigma}] = 4\eta_{\mu\nu} \eta^{\rho\alpha} \eta_{\alpha\sigma} = \boxed{4\eta_{\mu\nu} \delta^\rho_\sigma}$$

d) $\exp\left[i\frac{\pi}{2}\gamma^5\right]$

Solution [2 P]

$$\exp\left[i\frac{\pi}{2}\gamma^5\right] = \cos\left(\frac{\pi}{2}\right) + i\gamma^5 \sin\left(\frac{\pi}{2}\right) = \boxed{i\gamma^5}$$

Problem 2 (15 points)

Consider the following Lagrangian capturing the dynamics of a real scalar field ϕ in 4 space-time dimensions (we use units $c = \hbar = 1$)

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - c\phi^6 \ ,$$

with c a constant. The signature of Minkowski metric is mostly minus, i.e. $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$. Derive the corresponding interaction-picture Hamiltonian in terms of creation and annihilation operators.

Hint: The interaction-picture creation and annihilation operators $\hat{\alpha}_k^\dagger$ and $\hat{\alpha}_k$ are related to $\hat{a}_k^\dagger$ and $\hat{a}_k$ as

$$\hat{\alpha}_k^\dagger = \hat{a}_k^\dagger e^{i\omega_k t} \ , \quad \hat{\alpha}_k = \hat{a}_k e^{-i\omega_k t} \ .$$

Solution [5 (free part)+10(interaction part) P]

The Hamiltonian for the ϕ^6 theory given above can be found in a straightforward manner by writing the field in terms of creation and annihilation operators. Schematically, we find (box normalization, omitting δ-functions, and normal ordering)

$$H = \sum_k \int d^3\vec{x} \Big[\omega_k \hat{\alpha}_k^\dagger \hat{\alpha}_k + c\Big(90\hat{\alpha}_k^\dagger \hat{\alpha}_k + 60\hat{\alpha}_k^\dagger \hat{\alpha}_k^3 + 6\hat{\alpha}_k^\dagger \hat{\alpha}_k^5 + 90\hat{\alpha}_k^{\dagger 2} \hat{\alpha}_k^2 + 15\hat{\alpha}_k^{\dagger 2} \hat{\alpha}_k^4 + 60\hat{\alpha}_k^{\dagger 3} \hat{\alpha}_k + 20\hat{\alpha}_k^{\dagger 3} \hat{\alpha}_k^3 + 15\hat{\alpha}_k^{\dagger 4} \hat{\alpha}_k^2 + 6\hat{\alpha}_k^{\dagger 5} \hat{\alpha}_k \Big) \Big] \ .$$

Problem 3 (25 points)

Consider a theory given by the following Lagrangian density in 4 space-time dimensions (we use units $c = \hbar = 1$)

$$\mathcal{L} = \mathcal{L}_\varphi + \mathcal{L}_\psi + \mathcal{L}_{B_\mu} + \mathcal{L}_{\text{int}} \ ,$$

with

$$\begin{aligned}
\mathcal{L}_\varphi &= \frac{1}{2}(\partial_\mu \varphi)(\partial^\mu \varphi) - \frac{1}{2} M^2 \varphi^2 \ , \\
\mathcal{L}_\psi &= \bar{\psi}(i\partial\!\!\!/ - m)\psi \ , \\
\mathcal{L}_{B_\mu} &= -\frac{1}{4} G_{\mu\nu} G^{\mu\nu} + \frac{\mu^2}{2} B_\mu B^\mu \ , \\
\mathcal{L}_{\text{int}} &= -\lambda \varphi \bar{\psi}\psi - \frac{g}{2} \varphi B_\mu B^\mu \ ,
\end{aligned}$$

where φ is a massive real scalar field of mass M, ψ is a spinor of mass m with $\bar{\psi} \equiv \psi^\dagger \gamma^0$, B_μ is a massive vector field of mass μ, with its field strength-tensor given by $G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$, and λ and g are constants.

a) What are the mass dimensions of the following entities?

i) φ

ii) ψ

iii) M

iv) m

v) B_μ

vi) μ

vii) λ

viii) g

Solution [1 P]

$[S] \stackrel{!}{=} M^0 \quad \Leftrightarrow \quad [\mathcal{L}] = M^4$

i) $[\varphi] = M$ ii) $[\psi] = M^{3/2}$ iii) $[M] = M$ iv) $[m] = M$

v) $[B_\mu] = M$ vi) $[\mu] = M$ vii) $[\lambda] = M^0$ viii) $[g] = M$

b) Find the equations of motion.

Solution [3 P]

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} - \frac{\partial \mathcal{L}}{\partial \varphi} = 0 \quad \Leftrightarrow \quad \boxed{(\Box + M^2)\varphi + \lambda\bar{\psi}\psi + \frac{g}{2}B_\mu B^\mu = 0}$$

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} - \frac{\partial \mathcal{L}}{\partial \psi} = 0 \quad \Leftrightarrow \quad \boxed{i\partial_\mu\bar{\psi}\gamma^\mu + m\bar{\psi} + \lambda\varphi\bar{\psi} = 0}$$

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})} - \frac{\partial \mathcal{L}}{\partial \bar{\psi}} = 0 \quad \Leftrightarrow \quad \boxed{i\gamma^\mu\partial_\mu\psi - m\psi - \lambda\varphi\psi = 0}$$

$$\partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\nu B_\mu)} - \frac{\partial \mathcal{L}}{\partial B_\mu} = 0 \quad \Leftrightarrow \quad \boxed{\partial_\nu G^{\nu\mu} + \mu^2 B^\mu - g\varphi B^\mu = 0}$$

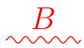
c) State the Feynman rules for all of the propagators and vertices of this theory. No derivation is necessary.

Solution [5 P]

Propagators:

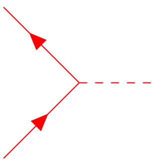
$$= \frac{i}{p^2 - M^2 + i\varepsilon}$$

$$= \frac{i}{\not{p} - m + i\varepsilon}$$

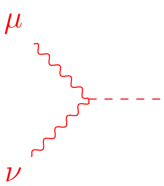


$$= \frac{-i\eta_{\mu\nu}}{p^2 - \mu^2 + i\varepsilon}$$

Vertices:



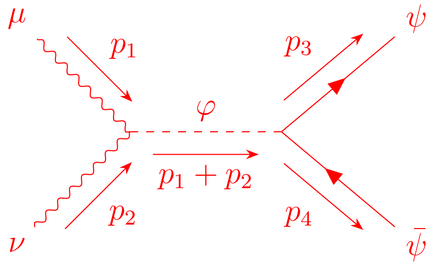
$$= -i\lambda$$



$$= -ig\eta^{\mu\nu}$$

d) Consider the process $BB \to \psi\bar{\psi}$. Draw and label the Feynman diagram(s) contributing to this process to leading order in λ and g.

Solution [1 P]



e) Under what conditions is the above process kinematically allowed?

Solution [1 P]

$2\mu > 2m \quad \Leftrightarrow \quad \boxed{\mu > m}$

f) How many physical polarizations does the massive vector boson B_μ have? Explain.

Solution [1 P]

3 physical polarizations; a massive vector boson in 4 space-time dimensions has 3 degrees of freedom.

g) Derive the spin-averaged amplitude squared for the above process in terms of the Mandelstam variables.

Hint: Take the polarization vectors $\varepsilon_\mu^{(i)}$ to be real and use the fact that the sum over the physical polarization states of a spin-1 particle of mass $\mu \neq 0$ is $\sum_i \varepsilon_\mu^{(i)} \varepsilon_\nu^{(i)} = -\eta_{\mu\nu} + \frac{k_\mu k_\nu}{\mu^2}$.

Solution [6 P]

For simplicity we abbreviate:

$u_i \equiv u(p_i, s_i)$, $v_i \equiv v(p_i, s_i)$ and $\varepsilon^\mu(p_1, i_1) \equiv \varepsilon_1^\mu$, $\varepsilon^\nu(p_2, i_2) \equiv \varepsilon_2^\nu$.

Below we will use $\sum_{s_i} u_i \bar{u}_i = \not{p}_i + m$ and $\sum_{s_i} v_i \bar{v}_i = \not{p}_i - m$, as well as the Mandelstam variable $s = (p_1 + p_2)^2 = (p_3 + p_4)^2$.

$$i\mathcal{M} = \varepsilon_1^\mu(-ig\eta_{\mu\nu})\varepsilon_2^\nu \frac{i}{s - M^2} \bar{u}_3(-i\lambda)v_4$$

$$\Leftrightarrow \quad \mathcal{M} = -\frac{g\lambda}{s - M^2} \varepsilon_1^\mu \varepsilon_2^\nu \eta_{\mu\nu} \bar{u}_3 v_4 \quad \Rightarrow \quad \mathcal{M}^\dagger = -\frac{g\lambda}{s - M^2} \varepsilon_1^\rho \varepsilon_2^\sigma \eta_{\rho\sigma} \bar{v}_4 u_3$$

$$\Rightarrow \quad \overline{|\mathcal{M}|^2} = \frac{1}{9} \sum_{i_1, i_2, s_3, s_4} \left(\frac{g\lambda}{s - M^2}\right)^2 \varepsilon_1^\mu \varepsilon_2^\nu \eta_{\mu\nu} \varepsilon_1^\rho \varepsilon_2^\sigma \eta_{\rho\sigma} \bar{u}_3 v_4 \bar{v}_4 u_3 =$$

$$= \left(\frac{g\lambda}{3(s - M^2)}\right)^2 \left(-\eta^{\mu\rho} + \frac{p_1^\mu p_1^\rho}{\mu^2}\right) \eta_{\mu\nu} \left(-\eta^{\nu\sigma} + \frac{p_2^\nu p_2^\sigma}{\mu^2}\right) \eta_{\rho\sigma} \mathrm{tr}[(\not{p}_3 + m)(\not{p}_4 - m)] =$$

$$= \left(\frac{g\lambda}{3(s - M^2)}\right)^2 \left(-\eta^{\mu\rho} + \frac{p_1^\mu p_1^\rho}{\mu^2}\right) \left(-\eta_{\mu\rho} + \frac{p_{2,\mu} p_{2,\rho}}{\mu^2}\right) 4(p_3 \cdot p_4 - m^2) =$$

$$= \left(\frac{2g\lambda}{3(s - M^2)}\right)^2 (p_3 \cdot p_4 - m^2) \left(4 - \frac{p_2^2}{\mu^2} - \frac{p_1^2}{\mu^2} + \frac{(p_1 \cdot p_2)^2}{\mu^4}\right)$$

Expressing $p_1 \cdot p_2$ and $p_3 \cdot p_4$ in terms of the Mandelstam variable s

$$p_1 \cdot p_2 = \frac{1}{2}(s - p_1^2 - p_2^2) = \frac{s}{2} - \mu^2\,, \quad p_3 \cdot p_4 = \frac{1}{2}(s - p_3^2 - p_4^2) = \frac{s}{2} - m^2\,,$$

we obtain:

$$\boxed{\overline{|\mathcal{M}|^2} = \left(\frac{2g\lambda}{3(s - M^2)}\right)^2 \left(\frac{s}{2} - 2m^2\right)\left(2 + \frac{(\frac{s}{2} - \mu^2)^2}{\mu^4}\right)}$$

h) Calculate the differential cross section of the process. Use the following general expression for the differential cross section of a 2-to-2 scattering process $AB \to CD$ in the center-of-mass (CM) frame:

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{CM}} = \frac{1}{2E_A 2E_B |\vec{v}_A - \vec{v}_B|} \frac{|\vec{p}_C|}{(2\pi)^2 4E_{\mathrm{CM}}} |\mathcal{M}(p_A, p_B \to p_C, p_D)|^2\,,$$

where E_A, E_B and E_{CM} are the energies of A, B and the total initial energy, respectively. Also, $\vec{v}_X = \dfrac{\vec{p}_X}{E_X}$ for $X = A, B$. Finally, $|\mathcal{M}(p_A, p_B \to p_C, p_D)|^2$ is the spin-averaged amplitude squared you found in g).

Solution [5 P]

Let $E = \dfrac{E_{\mathrm{CM}}}{2} = \dfrac{\sqrt{s}}{2}$

In the CM frame let

$$\begin{aligned} p_A = p_1 &= (E, 0, 0, |\vec{p}_1|) \\ p_B = p_2 &= (E, 0, 0, -|\vec{p}_1|) \\ p_C = p_3 &= (E, |\vec{p}_3|\mathrm{s}_\theta, 0, |\vec{p}_3|\mathrm{c}_\theta) \\ p_D = p_4 &= (E, -|\vec{p}_3|\mathrm{s}_\theta, 0, -|\vec{p}_3|\mathrm{c}_\theta) \end{aligned}$$

with $|\vec{p}_1| = \sqrt{E^2 - \mu^2}$ and $|\vec{p}_3| = \sqrt{E^2 - m^2}$.

$$\begin{aligned} \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{CM}} &= \frac{1}{4E^2 2\sqrt{E^2 - \mu^2}/E} \cdot \frac{\sqrt{E^2 - m^2}}{(2\pi)^2 4 \cdot 2E} \cdot \overline{|\mathcal{M}|^2} = \\ &= \sqrt{\frac{E^2 - m^2}{E^2 - \mu^2}} \frac{\overline{|\mathcal{M}|^2}}{4^3 (2\pi)^2 E^2} = \boxed{\sqrt{\frac{s/4 - m^2}{s/4 - \mu^2}} \frac{\overline{|\mathcal{M}|^2}}{(8\pi)^2 s}} \end{aligned}$$

i) Calculate the total cross section $\sigma = \dfrac{1}{N} \int \left(\dfrac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{CM}} \mathrm{d}\Omega$. What is the normalization factor N? Explain briefly.

Solution [2 P]

$N = 1$ as the particles in the final state are not identical.

$$\sigma = \int \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{CM}} \mathrm{d}\Omega = \int_0^{2\pi}\int_0^{\pi} \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{CM}} \mathrm{s}_\theta \, \mathrm{d}\theta \, \mathrm{d}\phi = 4\pi \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{CM}} = \boxed{\sqrt{\frac{s/4 - m^2}{s/4 - \mu^2}} \frac{\overline{|\mathcal{M}|^2}}{16\pi s}}$$

Problem 4 (15 points)

Consider the following action of a massless scalar field in $d > 2$ space-time dimensions

$$S = \int \mathrm{d}^d x \left[\frac{1}{2} (\partial_\mu \varphi)(\partial^\mu \varphi) - \lambda \varphi^{\frac{2d}{d-2}} \right] ,$$

with λ a real constant. The signature of Minkowski metric is mostly minus, i.e. $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, \dots)$.

a) What is the mass dimension of φ?

Solution [1 P]

$$[S] \overset{!}{=} M^0 \quad \Leftrightarrow \quad [\mathcal{L}] = M^d \quad \Rightarrow \quad \boxed{[\varphi] = M^{\frac{d}{2}-1}}$$

b) Find the equations of motion.

Solution [2 P]

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} - \frac{\partial \mathcal{L}}{\partial \varphi} = 0 \quad \Leftrightarrow \quad \boxed{\Box \varphi + \lambda \frac{2d}{d-2} \varphi^{\frac{d+2}{d-2}} = 0}$$

c) Consider the following transformation

$$\varphi(x) \to \varphi'(x) = \alpha^\Delta \varphi(\alpha x) \ .$$

Determine Δ such that the action be invariant under this. What do you observe?

Solution [3 P]

$$\begin{aligned} S \to S' &= \int \mathrm{d}^d x \left[\frac{1}{2} \left(\frac{\partial}{\partial x^\mu} \varphi'(x) \right)^2 - \lambda \varphi'^{\frac{2d}{d-2}} \right] = \\ &= \int \mathrm{d}^d x \left[\frac{1}{2} \alpha^{2(\Delta+1)} \left(\frac{\partial}{\partial(\alpha x^\mu)} \varphi(\alpha x) \right)^2 - \lambda \alpha^{\frac{2d\Delta}{d-2}} \varphi(\alpha x)^{\frac{2d}{d-2}} \right] \overset{y \equiv \alpha x}{=} \\ &= \int \mathrm{d}^d y \left[\frac{1}{2} \alpha^{2(\Delta+1)-d} \left(\partial_\mu \varphi(y) \right)^2 - \lambda \alpha^{\frac{2d\Delta}{d-2}-d} \varphi(y)^{\frac{2d}{d-2}} \right] \overset{!}{=} S \end{aligned}$$

$$\Leftrightarrow \quad \begin{cases} 2(\Delta+1) - d = 0 \\ \dfrac{2d\Delta}{d-2} - d = 0 \end{cases} \quad \Leftrightarrow \quad \boxed{\Delta = \frac{d}{2} - 1}$$

Δ is d-dependent. Invariance of kinetic and potential terms independently of each other gives $\Delta = \frac{d}{2} - 1$.

d) Find the corresponding Noether current. Show that it is conserved on the equations of motion.

Solution [9 P]

Let $\alpha = 1 + \varepsilon$, $\varepsilon \ll 1$

Then

$$\begin{aligned} \varphi'(x) &= (1+\varepsilon)^\Delta \varphi\left((1+\varepsilon)x\right) = \\ &= \left(1 + \varepsilon\Delta + \mathcal{O}(\varepsilon^2)\right) \left(\varphi(x) + \varepsilon x^\mu \partial_\mu \varphi(x) + \mathcal{O}(\varepsilon^2)\right) = \\ &= \varphi(x) + \varepsilon x^\mu \partial_\mu \varphi(x) + \varepsilon \Delta \varphi(x) + \mathcal{O}(\varepsilon^2) \end{aligned}$$

$\Rightarrow \delta\varphi = \varphi'(x) - \varphi(x) = \varepsilon\left(x^\mu \partial_\mu \varphi(x) + \Delta\varphi(x)\right) \equiv \varepsilon Q[\varphi]$
From Noether's theorem:

$$\begin{aligned} j^\mu &= \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)}\right) Q[\varphi] - x^\mu \mathcal{L} = \\ &= (\partial^\mu \varphi)\left(x^\nu \partial_\nu \varphi + (\frac{d}{2}-1)\varphi\right) - x^\mu \frac{1}{2}(\partial_\nu\varphi)(\partial^\nu\varphi) + x^\mu \lambda \varphi^{\frac{2d}{d-2}} \end{aligned}$$

$$\begin{aligned} \partial_\mu j^\mu &= (\Box\varphi)\left(x^\nu\partial_\nu\varphi + (\frac{d}{2}-1)\varphi\right) + (\partial^\mu\varphi)\left(\delta^\nu_\mu \partial_\nu\varphi + x^\nu \partial_\mu\partial_\nu\varphi + (\frac{d}{2}-1)\partial_\mu\varphi\right) \\ &- \frac{d}{2}(\partial_\nu\varphi)(\partial^\nu\varphi) - x^\mu(\partial_\mu\partial_\nu\varphi)(\partial^\nu\varphi) + d\lambda\varphi^{\frac{2d}{d-2}} + x^\mu\lambda\frac{2d}{d-2}\varphi^{\frac{d+2}{d-2}}(\partial_\mu\varphi) = \\ &= (\Box\varphi)\left(x^\nu\partial_\nu\varphi + (\frac{d}{2}-1)\varphi\right) + \left(x^\nu\partial_\nu\varphi + (\frac{d}{2}-1)\varphi\right)\lambda\frac{2d}{d-2}\varphi^{\frac{d+2}{d-2}} = \\ &= Q[\varphi]\underbrace{\left(\Box\varphi + \lambda\frac{2d}{d-2}\varphi^{\frac{d+2}{d-2}}\right)}_{=0 \text{ by EOM}} = 0 \quad \checkmark \end{aligned}$$

Problem 5 (25 points)

a) A Dirac spinor transforms under Lorentz transformation as $\psi(x) \mapsto \psi'(x') = S(\Lambda)\psi(x)$. Using the Lorentz invariance of the Dirac equation, show that

$$S(\Lambda) = \exp\left(-\frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}\right) .$$

Under infinitesimal Lorentz transformation the spacetime coordinates transform as

$$x^\mu \mapsto x'^\mu = \Lambda^\mu{}_\nu x^\nu = \left(\delta^\mu{}_\nu + \omega^\mu{}_\nu\right) x^\nu .$$

Recall that $\sigma^{\mu\nu}$ is defined by $\sigma^{\mu\nu} = \frac{i}{2}\left[\gamma^\mu, \gamma^\nu\right]$.

Hint 1: Note that $\omega^{\mu\nu}$ is anti-symmetric.
Hint 2: You will need to prove $\left[\gamma^\lambda, \sigma^{\mu\nu}\right] = 2i\left[\eta^{\lambda\mu}\gamma^\nu - \eta^{\lambda\nu}\gamma^\mu\right]$.

Solution [5 P]

Let us start with the proof of Hint 2. Using the identity $[A, BC] = \{A, B\}C - B\{A, C\}$ and the anti-commutation relation for the gamma matrices gives:

$$\begin{aligned} [\gamma^\lambda, \sigma^{\mu\nu}] &= \frac{i}{2}[\gamma^\lambda, \gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu] \\ &= \frac{i}{2}\left(\{\gamma^\lambda,\gamma^\mu\}\gamma^\nu - \gamma^\mu\{\gamma^\lambda,\gamma^\nu\} - \{\gamma^\lambda,\gamma^\nu\}\gamma^\mu + \gamma^\nu\{\gamma^\lambda,\gamma^\mu\}\right) \\ &= 2i\left(\eta^{\lambda\mu}\gamma^\nu - \eta^{\lambda\nu}\gamma^\mu\right) \end{aligned}$$

If we Lorentz transform the Dirac equation we obtain

$$
\begin{aligned}
\left(i\gamma^\mu \frac{\partial}{\partial x'^\mu} - m\right)\psi'(x') = 0 \\
\left(i\gamma^\mu \Lambda_\mu{}^\nu \frac{\partial}{\partial x^\nu} - m\right) S(\Lambda)\psi(x) = 0 \\
\left(iS^{-1}(\Lambda)\gamma^\mu S(\Lambda)\Lambda_\mu{}^\nu \frac{\partial}{\partial x^\nu} - m\right)\psi(x) = 0
\end{aligned}
$$

where in the last line we multiplied from left with $S^{-1}(\Lambda)$. To retain the original form of the Dirac equation, we should have

$$
S^{-1}(\Lambda)\gamma^\mu S(\Lambda) = \gamma^\nu (\Lambda^{-1})_\nu{}^\mu \tag{1}
$$

For infinitesimal Lorentz transformations we have

$$
\begin{aligned}
\gamma^\nu(\Lambda^{-1})_\nu{}^\mu &= \gamma^\nu(\delta_\nu{}^\mu - \omega_\nu{}^\mu) \\
&= \gamma^\mu - \frac{1}{2}(\gamma^\lambda \omega_\lambda{}^\mu - \gamma^\nu \omega^\mu{}_\nu) \\
&= \gamma^\mu - \frac{1}{2}\omega_{\lambda\nu}(\gamma^\lambda \eta^{\nu\mu} - \gamma^\nu \eta^{\lambda\mu}) \\
&= \gamma^\mu + \frac{i}{4}\omega_{\lambda\nu}[\sigma^{\lambda\nu}, \gamma^\mu]
\end{aligned} \tag{2}
$$

Note that in the last step we used the hint. In the infinitesimal case, $S(\Lambda)$ can be written as

$$
\begin{aligned}
S(\Lambda) &= 1 - \omega_{\lambda\nu} \left.\frac{\partial S(\Lambda)}{\partial \Lambda^\mu{}_\nu}\right|_{\Lambda^\mu{}_\nu} = \delta^\mu{}_\nu \\
&\equiv 1 - \omega_{\lambda\nu} G^{\lambda\nu}
\end{aligned}
$$

Inserting this into equation (1) gives up to the lowest order in ω

$$
S^{-1}(\Lambda)\gamma^\mu S(\Lambda) = \gamma^\mu + \omega_{\lambda\nu}[G^{\lambda\nu}, \gamma^\mu]
$$

Comparing with (2) we can identify $G^{\lambda\nu} = \frac{i}{4}\sigma^{\lambda\nu}$. Therefore we obtain

$$
S(\Lambda) = 1 - \frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu} \tag{3}
$$

for infinitesimal transformations and for finite transformation we get finally

$$
S(\Lambda) = e^{-\frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}} \tag{4}
$$

b) Since the Dirac Lagrangian

$$
\mathcal{L} = \bar{\psi}\left(i\gamma^\mu \partial_\mu - m\right)\psi \ ,
$$

is Lorentz invariant, there is a corresponding conserved Noether current. Using Noether's theorem show that the current can be written as

$$
J^\lambda = A_{\mu\nu} J^{\lambda,\mu\nu} \ ,
$$

where

$$J^{\lambda,\mu\nu} = i\bar{\psi}\gamma^{\lambda}\left(-\frac{i}{4}\sigma^{\mu\nu} + x^{\mu}\partial^{\nu}\right)\psi \ ,$$

and $A_{\mu\nu}$ is an arbitrary anti-symmetric tensor.

Solution [3 P]

The infinitesimal transformation for ψ is

$$\begin{aligned}\delta\psi(x) &= \psi'(x) - \psi(x)\\ &= \psi'(x' - \delta x) - \psi(x)\\ &= \psi'(x') - (\partial_{\mu}\psi'(x'))\delta x^{\mu} - \psi(x)\end{aligned}$$

With $\delta x^{\mu} = \omega^{\mu}{}_{\nu}x^{\nu}$ and $\psi'(x') - \psi(x) = -\frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}\psi(x)$ we obtain up to the first order in ω the following

$$\delta\psi(x) = \left(-\frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu} - \omega_{\mu\nu}x^{\nu}\partial^{\mu}\right)\psi(x)$$

Inserting this into the formula for the Noether current gives

$$J^{\lambda} = \frac{\partial\mathcal{L}}{\partial\partial_{\lambda}\psi}\delta\psi = iA_{\mu\nu}\bar{\psi}\gamma^{\lambda}\left(-\frac{i}{4}\sigma^{\mu\nu} + x^{\mu}\partial^{\nu}\right)\psi$$

where $A_{\mu\nu} = \omega_{\mu\nu}$. Note that the Lagrangian is not completely invariant under Lorentz transformations, because the transformation yields a total divergence which gives to the current an additional term of the form $A^{\lambda}{}_{\mu}x^{\mu}\mathcal{L}$. However, we can ignore this term in this exercise, because it is not relevant for the conservation of the current, because $A_{\mu\nu}$ is anti-symmetric. (To obtain the full points, both currents are correct)

c) Show that the current is conserved on the equations of motion.

Solution [3 P]

$$\begin{aligned}\partial_{\lambda}J^{\lambda} =& A_{\mu\nu}i(\partial_{\lambda}\bar{\psi})\gamma^{\lambda}\left(-\frac{i}{4}\sigma^{\mu\nu} + x^{\mu}\partial^{\nu}\right)\psi\\ &+ A_{\mu\nu}i\bar{\psi}\gamma^{\mu}\partial^{\nu}\psi\\ &+ A_{\mu\nu}i\bar{\psi}\left(-\frac{i}{4}\sigma^{\mu\nu} + x^{\mu}\partial^{\nu}\right)\gamma^{\lambda}\partial_{\lambda}\psi\\ &+ A_{\mu\nu}\bar{\psi}\frac{1}{4}[\gamma^{\lambda}, \sigma^{\mu\nu}]\partial_{\lambda}\psi\end{aligned}$$

Using the Dirac equation $i\gamma^{\mu}\partial_{\mu}\psi = m\psi$ and $i(\partial_{\mu}\bar{\psi})\gamma^{\mu} = -m\bar{\psi}$ one can show that the first and the third terms cancel. With the hint of part a) we obtain

$$\partial_{\lambda}J^{\lambda} = A_{\mu\nu}i\bar{\psi}\gamma^{\mu}\partial^{\nu}\psi + A_{\mu\nu}\left(\frac{i}{2}\bar{\psi}\gamma^{\nu}\partial^{\mu}\psi - \frac{i}{2}\bar{\psi}\gamma^{\mu}\partial^{\nu}\psi\right) = 0$$

where we used the anti-symmetry of $A_{\mu\nu}$.

d) Show that

$$J_k^\lambda = \varepsilon_{kij} J^{\lambda,ij} = \bar{\psi}\gamma^\lambda \left(S_k + L_k\right)\psi \ ,$$

where

$$S_k = \frac{\sigma_k}{2} \ , \qquad\qquad L_k = -i\varepsilon_{kij}x^i\partial_j \ .$$

Hint: Use the Dirac representation of the γ-matrices.

Solution [2 P]

Using the Dirac representation of the γ-matrices

$$\begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}$$

it is straightforward to show

$$\varepsilon_{kij}\sigma^{ij} = 2\sigma_k$$

With that we have

$$\begin{aligned} J_k^\lambda = \varepsilon_{kij} J^{\lambda,ij} &= i\bar{\psi}\gamma^\lambda \left(-\frac{i}{4}\varepsilon_{kij}\sigma^{ij} + \varepsilon_{kij}x^i\partial^j\right)\psi \\ &= \bar{\psi}\gamma^\lambda \left(\frac{\sigma_k}{2} - i\varepsilon_{kij}x^i\partial_j\right) \\ &= \bar{\psi}\gamma^\lambda(S_k + L_k)\psi \end{aligned}$$

e) The corresponding charge is given by

$$Q_k = \int \mathrm{d}^3x\ \psi^\dagger(\vec{x})\left(S_k + L_k\right)\psi(\vec{x}) \ .$$

Show that it satisfies the following equation

$$[Q_i, Q_j] = i\varepsilon_{ijk}Q_k \ .$$

Solution [5 P]

One can show explicitly that $[S_i, S_j] = i\varepsilon_{ijk}S_k$ and $[L_i, L_j] = i\varepsilon_{ijk}L_k$. However, to get the full points it is not necessary to show this, because this is a well-known relation in quantum mechanics. To show the commutation relation, we will need the following commutator

$$\begin{aligned} &\left[\psi_a^\dagger(\vec{x})\psi_b(\vec{x}), \psi_c^\dagger(\vec{y})\psi_d(\vec{y})\right] \\ &= \psi_c^\dagger(\vec{y})\left[\psi_a^\dagger(\vec{x})\psi_b(\vec{x}), \psi_d(\vec{y})\right] + \left[\psi_a^\dagger(\vec{x})\psi_b(\vec{x}), \psi_c^\dagger(\vec{y})\right]\psi_d(\vec{y}) \\ &= -\psi_c^\dagger(\vec{y})\{\psi_a^\dagger(\vec{x}), \psi_d(\vec{y})\}\psi_b(\vec{x}) + \psi_a^\dagger(\vec{x})\{\psi_b(\vec{x}), \psi_c^\dagger(\vec{y})\}\psi_d(\vec{y}) \end{aligned}$$

where in the last equality we used that $\{\psi, \psi\} = 0 = \{\psi^\dagger, \psi^\dagger\}$. For the other non-trivial anti-commutators we can use the Fourier-mode expansion

$$\begin{aligned}
\{\psi_a(\vec{x}), \psi_b^\dagger(\vec{y})\} &= \int \frac{\mathrm{d}^3 p}{(2\pi)^3 2\omega_p} \int \frac{\mathrm{d}^3 k}{(2\pi)^3 2\omega_k} \sum_i \sum_j \\
&\qquad \{\left(u_i^a(\vec{p}) a_i(\vec{p}) e^{-ipx} + v_i^a(\vec{p}) b_i^\dagger(\vec{p}) e^{ipx}\right), \left(u_j^{b\dagger}(\vec{k}) a_j^\dagger(\vec{k}) e^{iky} + v_j^{b\dagger}(\vec{k}) b_j(\vec{k}) e^{-iky}\right)\} \\
&= \int \frac{\mathrm{d}^3 p}{(2\pi)^3 2\omega_p} \int \frac{\mathrm{d}^3 k}{(2\pi)^3 2\omega_k} \sum_i \sum_j \\
&\qquad \left(u_i^a(\vec{p}) u_j^{b\dagger}(\vec{k}) (2\pi)^3 2\omega_k \delta_{ij} \delta^{(3)}(\vec{p} - \vec{k}) e^{-ipx + iky}\right. \\
&\qquad \left.+ v_i^a(\vec{p}) v_j^{b\dagger}(\vec{k}) (2\pi)^3 2\omega_k \delta_{ij} \delta^{(3)}(\vec{p} - \vec{k}) e^{ipx - iky}\right) \\
&= \int \frac{\mathrm{d}^3 p}{(2\pi)^3 2\omega_p} \sum_i \left(u_i^a(\vec{p}) u_i^{b\dagger} e^{i\vec{p}\cdot(\vec{x}-\vec{y})} + v_i^a(\vec{p}) v_i^{b\dagger}(\vec{p}) e^{-i\vec{p}\cdot(\vec{x}-\vec{y})}\right)
\end{aligned}$$

In the first step we used the anti-commutation relations for the creation and annihilation operators and in the second step we integrated over k and summed over j. Using the relations for the spinors from problem sheet 7, the sum over i can be evaluated

$$\begin{aligned}
\{\psi_a(\vec{x}), \psi_b^\dagger(\vec{y})\} &= \int \frac{\mathrm{d}^3 p}{(2\pi)^3 2\omega_p} \left(((\not{p} + m)\gamma^0)^{ab} e^{i\vec{p}\cdot(\vec{x}-\vec{y})} + ((\not{p} - m)\gamma^0)^{ab} e^{-i\vec{p}\cdot(\vec{x}-\vec{y})}\right) \\
&= \int \frac{\mathrm{d}^3 p}{(2\pi)^3 2\omega_p} \left(((\gamma^0 p^0 - \gamma^i p^i + m)\gamma^0)^{ab} + ((\gamma^0 p^0 + \gamma^i p^i - m)\gamma^0)^{ab}\right) e^{i\vec{p}\cdot(\vec{x}-\vec{y})} \\
&= \int \frac{\mathrm{d}^3 p}{(2\pi)^3 2\omega_p} 2p^0 \delta^{ab} e^{i\vec{p}\cdot(\vec{x}-\vec{y})} \\
&= \delta^{ab} \delta^{(3)}(\vec{x} - \vec{y})
\end{aligned}$$

Plugging this relation into the equation for the commutator from above give the following commutation relation

$$\left[\psi_a^\dagger(\vec{x})\psi_b(\vec{x}), \psi_c^\dagger(\vec{y})\psi_d(\vec{y})\right] = \psi_a^\dagger(\vec{x})\psi_d(\vec{y})\delta_{bc}\delta^{(3)}(\vec{x} - \vec{y}) - \psi_c^\dagger(\vec{y})\psi_b(\vec{x})\delta_{ad}\delta^{(3)}(\vec{x} - \vec{y}) \tag{5}$$

Using this we can analyse the commutation relation for Q_i:

$$\begin{aligned}
[Q_i, Q_j] &= \int \mathrm{d}^3 x \int \mathrm{d}^3 y \left[\psi^{\dagger ab}(\vec{x}) D_i^{ab} \psi^b(\vec{x}), \psi^{\dagger c}(\vec{y}) D_j^{cd} \psi^d(\vec{y})\right] \\
&= \int \mathrm{d}^3 x \int \mathrm{d}^3 y\; D_i^{ab} D_j^{cd} \left[\psi^{\dagger ab}(\vec{x}) \psi^b(\vec{x}), \psi^{\dagger c}(\vec{y}) \psi^d(\vec{y})\right]
\end{aligned}$$

where $D_i = S_i + L_i$. With (5) we get after integration

$$\begin{aligned}
[Q_i, Q_j] &= \int \mathrm{d}^3 x\; D_i^{ab} D_j^{cd} \left(\psi_a^\dagger(\vec{x})\psi_d(\vec{x})\delta_{bc} - \psi_c^\dagger(\vec{x})\psi_b(\vec{x})\delta_{ad}\right) \\
&= \int \mathrm{d}^3 x\; \psi^\dagger(\vec{x})[D_i, D_j]\psi(\vec{x}) \\
&= i\varepsilon_{ijk} \int \mathrm{d}^3 x\; \psi^\dagger(\vec{x}) D_k \psi(\vec{x}) \\
&= i\varepsilon_{ijk} Q_k
\end{aligned}$$

f) Instead of a Dirac field, consider now a real scalar field ϕ with Lagrangian density

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2 \ .$$

Show that the current J_k^λ for ϕ is

$$J_k^\lambda = -i \left(\partial^\lambda \phi\right) L_k \phi \ .$$

Solution [4 P]

The infinitesimal variation of the scalar field is

$$\begin{aligned}\delta\phi(x) &= \phi'(x) - \phi(x) \\ &= \phi'(x') - (\partial_\mu \phi'(x'))\delta x^\mu - \phi(x) \\ &= -(\partial_\mu \phi(x))\ \omega^\mu{}_\nu x^\nu\end{aligned}$$

where in the last equality we used the fact that for the scalar field $\phi'(x') = \phi(x)$. Plugging this expression into the formula for the Noether current gives

$$J^\lambda = \partial^\lambda \phi \ \omega_{\mu\nu} x^\mu \partial^\nu \phi$$

which leads to

$$J_k^\lambda = -i \partial^\lambda \phi L_k \phi$$

Note again that we ignored the total divergence term for the same reason as in b).

g) Interpret the above results. Compare the current associated with the spinor field to the corresponding scalar field current found in the previous point.

Solution [3 P]

The current/charge corresponds to angular momentum, which makes sense, because the spatial components of the Lorentz transformation matrix Λ describe rotations. Furthermore, we can observe that for the angular momentum of the Dirac field we have an additional spin contribution which is not there for the scalar field. Also this makes sense, because fermions described by ψ have spin$\frac{1}{2}$ and scalar bosons have spin 0.

Problem 6 (15 points)

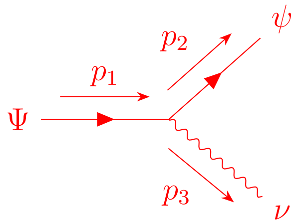
Consider the theory

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}\left(i\gamma^\mu \partial_\mu - m\right)\psi + \bar{\Psi}\left(i\gamma^\mu \partial_\mu - M\right)\Psi + g\bar{\psi}\gamma_\mu \Psi A^\mu \ .$$

Calculate the decay rate for the process $\Psi \to \psi\gamma$.
What are the conditions for this process to be kinematically allowed?

Solution [15 P]

Using the Feynman rules, the Feynman diagram for this decay and the corresponding amplitude are, in a self-explanatory notation, [3pt]



$$= -ig\bar{u}_2\gamma^\mu \bar{u}_1' \varepsilon_{3\mu} \,.$$

Then, we can calculate the squared amplitude and sum over the spins [5pt]

$$\begin{aligned}
\Rightarrow |\mathcal{M}|^2 &= -g^2 \left(\bar{u}_2\gamma^\mu u_1'^3 \varepsilon_{3\mu}\right)\left(\varepsilon_{3\nu}\bar{u'}_1\gamma^\nu u_2\right) \\
\Rightarrow \frac{1}{2}\sum_{s_1,s_2,s_3} |\mathcal{M}|^2 &= \frac{1}{2}\sum_{s_1,s_2} u_2\bar{u}_2\gamma^\mu u_1'^3 \sum_{s_3} \varepsilon_{3\mu}\varepsilon_{3\nu}\bar{u'}_1\gamma^\nu \\
&= -\frac{1}{2}g^2\,\mathrm{Tr}\left\{(\not{p}_2 + m)\gamma^\mu(\not{p}_1 + M)\gamma^\nu\right\} \\
&= -\frac{1}{2}g^2\left(-4p_1\cdot p_2 + 4mM\right) \\
&= g^2(M-m)^2
\end{aligned}$$

Finally, we calculate the decay rate [6pt]

$$\begin{aligned}
\Gamma &= \frac{1}{\omega_{\vec{p_1}}}\sum_{s_1,s_2}\int\frac{d^3\vec{p}_2}{(2\pi)^3 2\omega_{\vec{p_2}}}\int\frac{d^3\vec{p}_3}{(2\pi)^3 2\omega_{\vec{p_3}}}(2\pi)^4\delta^{(4)}\left(p_2+p_3-p_1\right) \\
&= \pi g^2\frac{(m-M)^2}{M}\int\frac{d^3\vec{p}_2}{(2\pi)^2 2\omega_{\vec{p_2}}}\frac{1}{2\omega_{\vec{p_3}}}\delta\left(\omega\vec{p}_2 + \omega_{\vec{p}_3} - M\right)
\end{aligned}$$

Working in the rest frame of Ψ and using the fact that $d|\vec{p}||\vec{p}| = d\omega\omega$, we are left with

$$\Rightarrow \Gamma = \frac{1}{8\pi}g^2\frac{(m-M)^2}{M}\int\frac{d\omega_{\vec{p}_2}\omega_{\vec{p}_2}}{\omega_{p_0}}\delta\left(\omega_{p_2} + |\vec{p}_2| - M\right)$$

Now we can switch coordinates to $u = \omega_{p_2} + |\vec{p}_2|$ to perform the integral

$$d\omega_{\vec{p}_2} = \frac{1}{2}\left(1 - \frac{m^2}{\mu^2}\right)du$$

$$\begin{aligned}
\Rightarrow \Gamma &= \frac{1}{16\pi}g^2\frac{(m-M)^2}{M}\int du\delta(u-M)\left(1-\frac{m^2}{u^2}\right) \\
&= \frac{1}{16\pi}g^2\frac{(m-M)^2}{M}\frac{M^2-m^2}{M^2} = g^2\left(\frac{M-m}{M}\right)^3\frac{M+m}{16\pi}
\end{aligned}$$

In order for the process to be kinematically allowed, we must have $M > m$ [1pt]

Problem 7 (20 points)

Consider the following action

$$S = \frac{1}{2}\int \mathrm{d}^4x\left[(\partial_\mu\varphi_1)(\partial^\mu\varphi_1) + (\partial_\mu\varphi_2)(\partial^\mu\varphi_2) + \frac{2\varphi_2}{\lambda}(\partial_\mu\varphi_1)(\partial^\mu\varphi_2) + \frac{\varphi_2^2}{\lambda^2}(\partial_\mu\varphi_2)(\partial^\mu\varphi_2)\right] ,$$

where φ_1 and φ_2 are two real massless scalar fields.

a) What is the mass dimension of λ?

Solution [5 P]

$[\lambda] = M.$

b) Write down the Feynman rules.

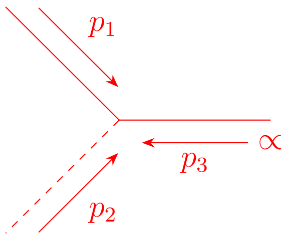
Solution [5 P]

Propagators:

$$\varphi_1 = \frac{i}{p^2 + i\varepsilon}$$

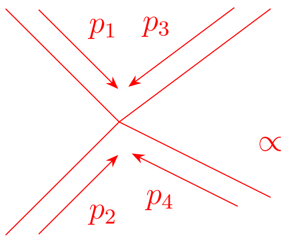
$$\varphi_2 = \frac{i}{p^2 + i\varepsilon}$$

Vertices:



$$\propto \frac{p_2 \cdot (p_1 + p_3)}{\lambda}$$

Note that in the process in c) you can use for the momenta of the off-shell particles the momentum conservation, such that this vertex factor is proportional to p_2^2.



$$\propto \frac{p_1^2 + p_2^2 + p_3^2 + p_1 p_2 + p_1 p_3 + p_2 p_3}{\lambda^2}$$

c) Calculate the lowest order (in $1/\lambda^2$) Feynman amplitude for the process $\varphi_1\varphi_1 \to \varphi_2\varphi_2$.

Hint: You have to consider two Feynman diagrams.

Solution [5 P]

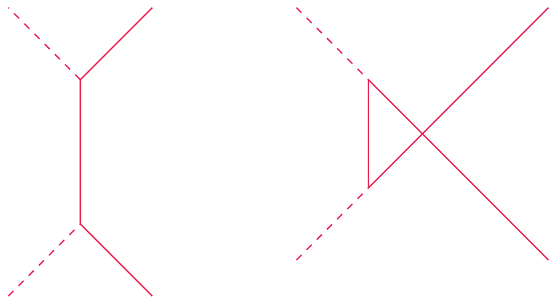
The amplitude for this process involves two Feynman diagrams—one in the t and the other in the u channel–both with a φ_2 internal line:

Using the Feynman rules, we find easily that the diagrams sum up to

$$\mathcal{M} = 0 \, .$$

d) Use the redefinition

$$\chi_1 = \varphi_1 + \frac{\varphi_2^2}{2\lambda}, \qquad \chi_2 = \varphi_2 \, ,$$



to rewrite the action.

What do you observe? With this new insight, interpret the result of point c).

Solution [5 P]

First we invert the above relations to express $\varphi_{1,2}$ in terms of $\chi_{1,2}$; we easily find

$$\varphi_1 = \chi_1 - \frac{\chi_2^2}{2\lambda} , \quad \varphi_2 = \chi_2 . \tag{6}$$

Plugging this into the action we find

$$S = \frac{1}{2} \int \mathrm{d}^4x \left[(\partial_\mu \chi_1)(\partial^\mu \chi_1) + (\partial_\mu \chi_2)(\partial^\mu \chi_2) \right] , \tag{7}$$

which is nothing more than two massless, completely decoupled real scalars. Therefore, the fact that the amplitude is zero is of course expected — using the $\chi_{1,2}$ fields we wrote down a simple action in a complicated manner.