Ludwig-Maximilians-Universität München

Quantum Field Theory (Quantum Electrodynamics)

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Guidelines:

- The exam consists of 6 problems.
- The duration of the exam is 48 hours.
- Please write your name or matriculation number on every sheet that you hand in.
- Your answers should be comprehensible and readable.

GOOD LUCK!

Exercise 1	10 P
Exercise 2	15 P
Exercise 3	30 P
Exercise 4	20 P
Exercise 5	20 P
Exercise 6	20 P

Total	115 P

Problem 1 (10 points)

How many independent real scalar fields enter the following Lagrangians? Justify your answers by explicit computations. Assume φ and χ to be real scalar fields.

Hint: You may find the non-zero eigenvalues of the kinetic and mass matrices.

a)

$$\mathcal{L}_1 = \frac{1}{2}(\partial_\mu \chi)^2 + 2(\partial_\mu \varphi)^2 + 2(\partial_\mu \chi)(\partial^\mu \varphi) \ .$$

Solution [5 P]

The kinetic matrix of the above reads

$$\begin{pmatrix} \frac{1}{2} & 1 \\ 1 & 2 \end{pmatrix} ,$$

whose determinant is zero. This means that there is only one field.

b)

$$\mathcal{L}_2 = \mathcal{L}_1 + \frac{1}{2}(\partial_\mu \chi)^2 + \frac{3}{2}(\partial_\mu \varphi)^2 + (\partial_\mu \chi)(\partial^\mu \varphi) - m^2\chi^2 - 2m^2\varphi^2 \ .$$

Solution [5 P]

The kinetic matrix of the above reads

$$\begin{pmatrix} 1 & \frac{3}{2} \\ \frac{3}{2} & \frac{7}{2} \end{pmatrix} ,$$

whose determinant is not zero. This means that there are two fields.

Problem 2 (15 points)

a) Take the following Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)^2 + a\varphi + b\varphi^2 + c\varphi^3 + d\varphi^4 \ ,$$

with φ a real scalar field and a, b, c, d constants. Assume invariance under the discrete transformation $\varphi \to -\varphi$. What can you say about the parameters a, b, c, d?

Solution [5 P]

Invariance under $\varphi \to -\varphi$ dictates that

$$a = 0 \ , \quad c = 0 \ .$$

b) Take the following Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi^*)(\partial^\mu \varphi) + a(\varphi + \varphi^*) + b(\varphi^2 + \varphi^{*2}) + c(\varphi^3 + \varphi^{*3}) + d(\varphi\varphi^*)^2 \ ,$$

with φ a complex scalar field and a, b, c, d constants. Assume invariance under the discrete transformation $\varphi \to e^{\frac{i2\pi N}{3}}\varphi, N =$ integer. What can you say about the parameters a, b, c, d?

Solution [5 P]

Invariance under $\varphi \to e^{\frac{i2\pi N}{3}}\varphi$ dictates that

$$a = 0 \; , \quad b = 0 \; .$$

c) The Lagrangian describing fermions interacting with photons is

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} + e_1\bar{\psi}\gamma_\mu\psi A^\mu + e_2\bar{\psi}\gamma_\mu\gamma^5\psi A^\mu \; ,$$

where ψ is a fermion field, $\bar{\psi} = \psi^\dagger\gamma^0$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. If A^μ is a vector under parity and parity is a symmetry of the Lagrangian, what can you say about the coefficients e_1 and e_2?

Solution [5 P]

Requiring that the Lagrangian is invariant under parity, dictates that

$$e_2 = 0 \; .$$

This is aftermath of the fact that the quantity

$$\bar{\psi}\gamma_\mu\gamma^5\psi \; ,$$

is a pseudo-vector. At the same time,

$$\bar{\psi}\gamma_\mu\psi \; ,$$

is a vector.

Problem 3 (30 points)

Consider a theory given by the following Lagrangian in 4 space-time dimensions (we use units $c = \hbar = 1$)

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\varphi)(\partial^\mu\varphi) - \frac{1}{2}M^2\varphi^2 + \bar{\psi}(i\partial\!\!\!/ - m)\psi - \frac{\lambda}{2}\varphi^2\bar{\psi}\psi \; ,$$

where φ is a real scalar field of mass M, ψ is a Dirac spinor of mass m with $\bar{\psi} \equiv \psi^\dagger\gamma^0$, and λ is a coupling constant.

a) What are the mass dimensions of the following entities?

i) ψ

ii) φ

iii) λ

Solution [3 P]

$[\mathcal{L}] \stackrel{!}{=} M^4$

i) $[\mathcal{L}] = M[\psi]^2 \quad \Leftrightarrow \quad \boxed{[\psi] = M^{3/2}}$

ii) $[\mathcal{L}] = M^2[\varphi]^2 \quad \Leftrightarrow \quad \boxed{[\varphi] = M}$

iii) $[\mathcal{L}] = [\lambda][\varphi]^2[\psi]^2 \quad \Leftrightarrow \quad \boxed{[\lambda] = M^{-1}}$

b) Find the equations of motion.

Solution [3 P]

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} - \frac{\partial \mathcal{L}}{\partial \psi} = 0 \quad \Leftrightarrow \quad \boxed{i\partial_\mu \bar{\psi}\gamma^\mu + m\bar{\psi} + \frac{\lambda}{2}\varphi^2\bar{\psi} = 0}$$
$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})} - \frac{\partial \mathcal{L}}{\partial \bar{\psi}} = 0 \quad \Leftrightarrow \quad \boxed{i\gamma^\mu\partial_\mu \psi - m\psi - \frac{\lambda}{2}\varphi^2\psi = 0}$$
$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} - \frac{\partial \mathcal{L}}{\partial \varphi} = 0 \quad \Leftrightarrow \quad \boxed{\partial_\mu\partial^\mu\varphi + M^2\varphi + \lambda\varphi\bar{\psi}\psi = 0}$$

c) Check explicitly whether the theory is invariant under the following transformations

i) $\psi \to \psi' = \mathrm{e}^{-i\alpha}\psi\,,$

ii) $\psi \to \psi'' = e^{-i\beta\gamma_5}\psi\,,$

with α, β constants. If yes, derive the corresponding Noether currents and show that they are conserved on the equations of motion.

Solution [8 P]

i) $U \equiv \mathrm{e}^{-i\alpha} = const$

$\bar{\psi}'\psi' = (U\psi)^\dagger\gamma_0 U\psi = \psi^\dagger U^\dagger\gamma_0 U\psi = \psi^\dagger\gamma_0 \underbrace{U^\dagger U}_{=I}\psi = \bar{\psi}\psi \quad \checkmark \quad \Rightarrow \quad$ invariant

$$\bar{\psi}'(i\partial\!\!\!/ - m)\psi' = (U\psi)^\dagger\gamma_0(i\partial\!\!\!/ - m)U\psi = \psi^\dagger U^\dagger\gamma_0(i\partial\!\!\!/ - m)U\psi = \\ = \psi^\dagger\gamma_0\underbrace{U^\dagger U}_{=I}(i\partial\!\!\!/ - m)\psi = \bar{\psi}(i\partial\!\!\!/ - m)\psi \quad \checkmark \quad \Rightarrow \quad \text{invariant}$$

$\psi \to \psi' = \mathrm{e}^{-i\alpha}\psi \stackrel{\alpha \lll 1}{=} (1 - i\alpha)\psi = \psi - i\alpha\psi \stackrel{!}{=} \psi + \alpha\delta\psi \quad \Leftrightarrow \quad \delta\psi = -i\psi$

Noether current: $J^\mu = \dfrac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\delta\psi = i\bar{\psi}\gamma^\mu(-i\psi) = \boxed{\bar{\psi}\gamma^\mu\psi}$

$$\partial_\mu J^\mu = (\partial_\mu\bar{\psi})\gamma^\mu\psi + \bar{\psi}\gamma^\mu(\partial_\mu\psi) \stackrel{e.o.m.\ in\ b)}{=} \\ = (im\bar{\psi} + i\frac{\lambda}{2}\varphi^2\bar{\psi})\psi + \bar{\psi}(-im\psi - i\frac{\lambda}{2}\varphi^2\psi) = 0 \quad \checkmark \quad \Rightarrow \quad \text{conserved}$$

ii) $U \equiv \mathrm{e}^{-i\beta\gamma_5} = const$

First, consider

$$U^\dagger\gamma_0 \overset{\gamma_5^\dagger=\gamma_5}{=} \mathrm{e}^{i\beta\gamma_5}\gamma_0 = \sum_{n=0}^{\infty}\frac{(i\beta\gamma_5)^n}{n!}\gamma_0 \overset{(\gamma_5)^2=1}{=} (\cos\beta + i\gamma_5\sin\beta)\gamma_0 =$$
$$\overset{\{\gamma_0,\gamma_5\}=1}{=} \gamma_0(\cos\beta - i\gamma_5\sin\beta) = \gamma_0\mathrm{e}^{-i\beta\gamma_5} = \gamma_0 U$$

and

$UU = \mathrm{e}^{-2i\beta\gamma_5} \neq I$

Then

$\bar{\psi}''\psi'' = (U\psi)^\dagger\gamma_0 U\psi = \psi^\dagger U^\dagger\gamma_0 U\psi = \psi^\dagger\gamma_0 \underbrace{UU}_{\neq I}\psi \neq \bar{\psi}\psi \quad \Rightarrow \quad$ not invariant

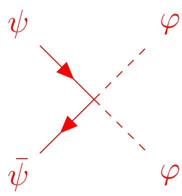
Therefore, there is also no corresponding Noether current.

d) State the Feynman rules for all of the propagators and vertices of this theory. No derivation is necessary.

Solution [3 P]

The required Feynman rules in momentum space:

Vertex:



$= -i\lambda$

Real scalar propagator:

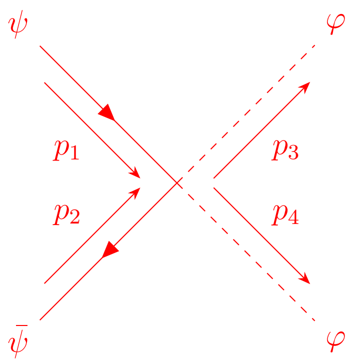
$$= \frac{i}{p^2 - M^2 + i\varepsilon}$$

Spinor propagator:

$$= \frac{i}{\not{p} - m + i\varepsilon}$$

e) Consider the process $\psi\bar{\psi} \to \varphi\varphi$. Draw and label the Feynman diagram(s) contributing to this process to leading order in λ.

Solution [3 P]



f) Under what conditions is the above process kinematically allowed?

Solution [1 P]

$2m > 2M \quad \Leftrightarrow \quad \boxed{m > M}$

g) Derive the spin-averaged amplitude squared for the above process in terms of the Mandelstam variables.

Solution [9 P]

For simplicity we abbreviate $u_i \equiv u(p_i, s_i)$ and $v_i \equiv v(p_i, s_i)$.

$\mathcal{M} = \bar{v}_2(-\lambda)u_1 \quad \Rightarrow \quad \mathcal{M}^\dagger = -\lambda \bar{u}_1 v_2$

Below we will use $\sum_{s_i} u_i\bar{u}_i = \not{p}_i + m$ and $\sum_{s_i} v_i\bar{v}_i = \not{p}_i - m$

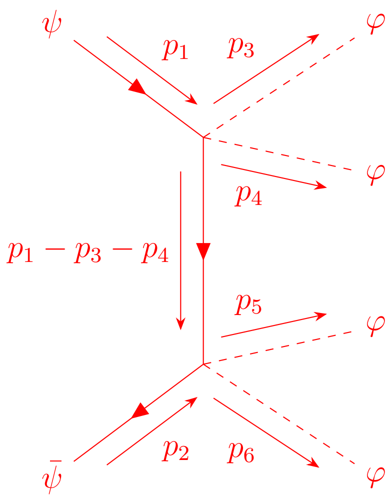
$$\frac{1}{4}\sum_{\{s\}}|\mathcal{M}|^2 = \frac{\lambda^2}{4}\sum_{s_1,s_2}\bar{v}_2u_1\bar{u}_1v_2 = \frac{\lambda^2}{4}\text{tr}[(\not{p}_1 + m)(\not{p}_2 - m)] = \lambda^2(p_1\cdot p_2 - m^2)$$

Using Mandelstam variables: $p_1 \cdot p_2 = \frac{1}{2}(s - p_1^2 - p_2^2) = \frac{s}{2} - m^2$

$$\Rightarrow \frac{1}{4}\sum_{\{s\}}|\mathcal{M}|^2 = \frac{\lambda^2}{2}(s - 4m^2)$$

Bonus (extra 5 pts): Draw and label an example of a tree-level Feynman diagram that is second order in λ describing the process $\psi\bar{\psi} \to \varphi\varphi\varphi\varphi$.

Solution [5 P (Bonus)]



Problem 4 (20 points)

Consider a theory given by the following Lagrangian in 4 space-time dimensions (we use units $c = \hbar = 1$)

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi - e\bar{\psi}\gamma^\mu B_\mu \psi - \frac{1}{4}G_{\mu\nu}G^{\mu\nu} + \frac{\mu^2}{2}B_\mu B^\mu + (D_\mu\varphi)^*(D^\mu\varphi) - M^2\varphi^*\varphi \ ,$$

where ψ is a Dirac spinor of mass m with $\bar{\psi} \equiv \psi^\dagger\gamma^0$, B_μ a massive vector field of mass μ, with its field strength-tensor given by $G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$, and φ a massive complex scalar field of mass M. Also, D_μ is defined as $D_\mu = \partial_\mu + igB_\mu$. Finally, e and g are constants.

a) Find the equations of motion.

Solution [6 P]

$$\partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)} - \frac{\partial\mathcal{L}}{\partial\psi} = 0 \quad \Leftrightarrow \quad \boxed{i\partial_\mu\bar{\psi}\gamma^\mu + m\bar{\psi} + e\bar{\psi}\gamma^\mu B_\mu = 0}$$

$$\partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})} - \frac{\partial\mathcal{L}}{\partial\bar{\psi}} = 0 \quad \Leftrightarrow \quad \boxed{i\gamma^\mu\partial_\mu\psi - m\psi - e\gamma^\mu B_\mu\psi = 0}$$

$$\begin{aligned}\mathcal{L} \supset (D_\mu\varphi)^*(D^\mu\varphi) &= (\partial_\mu\varphi^* - igB_\mu\varphi^*)(\partial^\mu\varphi + igB_\mu\varphi) = \\ &= (\partial_\mu\varphi^*)(\partial^\mu\varphi) + igB_\mu[\varphi(\partial_\mu\varphi^*) - (\partial^\mu\varphi)\varphi^*] + g^2B_\mu B^\mu\varphi^*\varphi\end{aligned}$$

$$\partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)} - \frac{\partial\mathcal{L}}{\partial\varphi} = 0$$

$$\Leftrightarrow \quad \boxed{(\partial_\mu\partial^\mu + M^2)\varphi^* = ig\partial_\mu(B^\mu\varphi^*) + igB_\mu(\partial^\mu\varphi^*) + g^2B_\mu B^\mu\varphi^*}$$

$$\partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi^*)} - \frac{\partial\mathcal{L}}{\partial\varphi^*} = 0$$

$$\Leftrightarrow \quad \boxed{(\partial_\mu\partial^\mu + M^2)\varphi = -ig\partial_\mu(B^\mu\varphi) - igB_\mu(\partial^\mu\varphi) + g^2B_\mu B^\mu\varphi}$$

$$\partial_\nu \frac{\partial\mathcal{L}}{\partial(\partial_\nu B_\mu)} - \frac{\partial\mathcal{L}}{\partial B_\mu} = 0$$

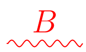
$$\Leftrightarrow \quad \boxed{\partial_\nu G^{\nu\mu} + \mu^2B^\mu - e\bar{\psi}\gamma^\mu\psi + ig[\varphi(\partial^\mu\varphi^*) - (\partial^\mu\varphi)\varphi^*] + 2g^2B^\mu\varphi^*\varphi = 0}$$

b) State the Feynman rules for all of the propagators and vertices of the theory. No derivation is necessary.

Solution [6 P]

Propagators:

$$= \frac{i}{p^2 - M^2 + i\varepsilon}$$



$$= \frac{-i\eta_{\mu\nu}}{p^2 - \mu^2 + i\varepsilon}$$

$$= \frac{i}{p\!\!\!/ - m + i\varepsilon}$$

Vertices:

$S^{int}_{B\varphi\varphi^*} = ig \int \mathrm{d}^4x\, B_\mu[\varphi(\partial^\mu\varphi^*) - (\partial^\mu\varphi)\varphi^*]$

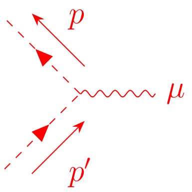
Fourier-transform the fields:

$$B^\mu(x) = \int \frac{\mathrm{d}^4k}{(2\pi)^4} \mathrm{e}^{-ikx} \tilde{B}^\mu(k)$$
$$\varphi(x) = \int \frac{\mathrm{d}^4p}{(2\pi)^4} \mathrm{e}^{-ipx} \tilde{\varphi}(p)$$

Therefore

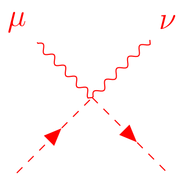
$$\begin{aligned} S^{int}_{B\varphi\varphi^*} &= ig \int \mathrm{d}^4x \int \frac{\mathrm{d}^4k}{(2\pi)^4} \mathrm{e}^{-ikx} \tilde{B}_\mu(k) \int \frac{\mathrm{d}^4p}{(2\pi)^4} \int \frac{\mathrm{d}^4p'}{(2\pi)^4} \mathrm{e}^{-ipx} \mathrm{e}^{ip'x} \cdot \\ &\cdot [\tilde{\varphi}(p)(ip'^\mu)\tilde{\varphi}^*(p') - (-ip^\mu)\tilde{\varphi}(p)\tilde{\varphi}^*(p')] = \\ &= -g \int \frac{\mathrm{d}^4p}{(2\pi)^4} \int \frac{\mathrm{d}^4p'}{(2\pi)^4} \tilde{B}_\mu(p'-p)\tilde{\varphi}(p)\tilde{\varphi}^*(p')(p+p')^\mu \end{aligned}$$

Due to the expansion of $\mathrm{e}^{iS^{int}_{B\varphi\varphi^*}}$ the vertex acquires an extra i factor. Therefore



$= -ig(p+p')^\mu$

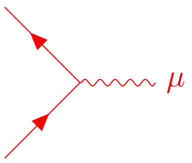
For the quartic vertex we obtain



$= 2ig^2\eta^{\mu\nu}$

where the factor 2 comes from the two possible contractions of B_μ.

Finally, the other cubic vertex is identical to that in QED



$= -ie\gamma^\mu$

c) Let ψ transform under a local $U(1)$ symmetry: $\psi \to \psi' = \mathrm{e}^{-i\alpha}\psi$, where $\alpha = \alpha(x)$. Derive explicitly how the other fields have to transform, and what conditions the parameters m, M, μ, e, g have to satisfy, so that the theory be invariant under this transformation.

Hint: Consider first the invariance of $\bar{\psi}(i\partial\!\!\!/ - m)\psi - e\bar{\psi}\gamma^\mu B_\mu\psi$, then the invariance of $(D_\mu\varphi)^*(D^\mu\varphi) - M^2\varphi^*\varphi$, and finally the invariance of the remaining terms.

Solution [6 P]

$$\bar{\psi}'(i\not{\partial} - m)\psi' - e\bar{\psi}'\gamma^\mu B'_\mu\psi' = \bar{\psi}(i\not{\partial} - m)\psi + \bar{\psi}i\gamma^\mu(-i\partial_\mu\alpha)\psi - e\bar{\psi}\gamma^\mu B'_\mu\psi =$$
$$\stackrel{!}{=} \bar{\psi}(i\not{\partial} - m)\psi - e\bar{\psi}\gamma^\mu B_\mu\psi$$
$$\Leftrightarrow \quad \bar{\psi}\gamma^\mu(\partial_\mu\alpha)\psi - e\bar{\psi}\gamma^\mu B'_\mu\psi = -e\bar{\psi}\gamma^\mu B_\mu\psi$$
$$\Leftrightarrow \quad \boxed{B'_\mu = B_\mu + \frac{1}{e}\partial_\mu\alpha}$$

For the covariant derivative $D_\mu\varphi$ to transform under some local $U(1)$ symmetry, $D_\mu\varphi$ and φ have to transform in the same way: $\varphi \to \varphi' = \mathrm{e}^{-i\beta}\varphi$ for some local $U(1)$ transformation β.

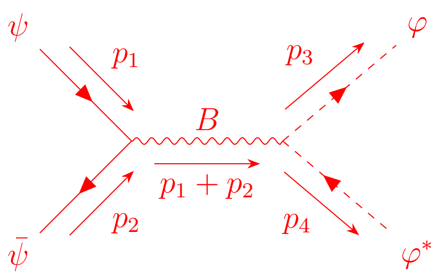
$$D'_\mu\varphi' \stackrel{!}{=} \mathrm{e}^{-i\beta}D_\mu\varphi$$
$$\Leftrightarrow \quad [\partial_\mu + ig(B_\mu + \frac{1}{e}\partial_\mu\alpha)]\mathrm{e}^{-i\beta}\varphi \stackrel{!}{=} \mathrm{e}^{-i\beta}[\partial_\mu + igB_\mu]\varphi$$
$$\Leftrightarrow \quad \mathrm{e}^{-i\beta}[-i(\partial_\mu\beta) + \partial_\mu + igB_\mu + i\frac{g}{e}(\partial_\mu\alpha)]\varphi \stackrel{!}{=} \mathrm{e}^{-i\beta}[\partial_\mu + igB_\mu]\varphi$$
$$\Leftrightarrow \quad -i(\partial_\mu\beta) + i\frac{g}{e}(\partial_\mu\alpha) = 0$$
$$\Leftrightarrow \quad \beta = \frac{g}{e}\alpha$$
$$\Leftrightarrow \quad \boxed{\varphi' = \exp\left[-i\frac{g}{e}\alpha\right]\varphi}$$

$$\varphi^{*\prime}\varphi' = \varphi^*\exp\left[i\frac{g}{e}\alpha\right]\exp\left[-i\frac{g}{e}\alpha\right]\varphi = \varphi^*\varphi \quad \checkmark$$
$$G'_{\mu\nu} = \partial_\mu B'_\nu - \partial_\nu B'_\mu = \partial_\mu B_\nu + \frac{1}{e}\partial_\mu\partial_\nu\alpha - \partial_\nu B_\mu - \frac{1}{e}\partial_\nu\partial_\mu\alpha = G_{\mu\nu} \quad \checkmark$$

$$\frac{\mu^2}{2}B'_\mu B^{\mu\prime} = \frac{\mu^2}{2}(B_\mu + \frac{1}{e}\partial_\mu\alpha)(B^\mu + \frac{1}{e}\partial^\mu\alpha) =$$
$$= \frac{\mu^2}{2}[B_\mu B^\mu + \frac{2}{e}B_\mu(\partial^\mu\alpha) + \frac{1}{e^2}(\partial_\mu\alpha)(\partial^\mu\alpha)] \stackrel{!}{=} \frac{\mu^2}{2}B_\mu B^\mu$$
$$\Leftrightarrow \quad \boxed{\mu = 0}$$

d) Draw and label an example of a Feynman diagram describing the process $\psi\bar{\psi} \to \varphi\varphi^*$.

Solution [2 P]



Problem 5 (20 points)

a) Show that for the scattering of an electron $e^-(p) \to e^-(p')$ off an external static potential $A_\mu(0, \vec{x})$

$$\langle f|\, S - 1\,|i\rangle = i\mathcal{M}\, 2\pi\delta(E' - E) \equiv ie\, 2\pi\delta(E' - E)\, \bar{u}(p')\gamma^\mu u(p)\tilde{A}_\mu(\vec{q}) \;,$$

where $q = p' - p$ and $\tilde{A}_\mu(\vec{q}) = \int \mathrm{d}^3x\, \mathrm{e}^{-i\vec{q}\cdot\vec{x}} A_\mu(\vec{x})$ is the Fourier transform of the potential.

Hint: A^μ is an external classical c-number, so it is not involved in any contractions.

Solution [6 P]

At first order in e, we have

$$\begin{aligned}
\langle f|S-1|i\rangle &= ie\int d^4x \langle f|\bar{\psi}(x)\gamma_\mu\psi(x)A^\mu(\vec{x})|i\rangle \\
&= ie\int d^4x \langle 0|a(p')\bar{\psi}(x)\gamma_\mu\psi(x)A^\mu(\vec{x})a^\dagger(p)|0\rangle \\
&= ie\int d^4x \bar{u}(p')\gamma_\mu u(p)A^\mu(\vec{x})e^{i(p'-p)x} \\
&= ie\int dx^0 e^{i(E'-E)x^0}\bar{u}(p')\gamma_\mu u(p)\int d^3\vec{x}A^\mu(\vec{x})e^{i(\vec{p}'-\vec{p})\vec{x}} \\
&= ie2\pi\delta(E'-E)\bar{u}(p')\gamma_\mu u(p)\tilde{A}^\mu(\vec{q}) \;.
\end{aligned}$$

b) Next derive the following expression for the cross section (β is the velocity of the incoming electron)

$$\mathrm{d}\sigma = |\mathcal{M}|^2\, 2\pi\delta(E' - E)\frac{1}{2E\beta}\frac{\mathrm{d}^3p'}{(2\pi)^3 2E'} \;.$$

Solution [6 P]

We start from the definition of the differential cross section

$$d\sigma = \frac{V}{T\beta}\frac{|\langle f|S-1|i\rangle|^2}{2EV}\frac{V d^3\vec{p}'}{(2\pi)^3 2E'V} \;.$$

Plugging what we found from point *a)* and after some straightforward massaging we obtain the desired result.

c) With the above equation calculate

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha^2}{4|\vec{p}|^2\beta^2\sin^4\frac{\theta}{2}}\left(1 - \beta^2\sin^2\frac{\theta}{2}\right) \;,$$

for an electron scattering off the following potential $A_\mu(\vec{x}) = (\frac{e}{4\pi|\vec{x}|}, 0, 0, 0)$. In the above $\alpha = e^2/4\pi$ is the fine-structure constant and θ is the angle between $\vec{p}$ and $\vec{p}'$.

Solution [8 P]

Using the explicit form of the gauge potential, we compute the spin-averaged amplitude squared to find

$$|\bar{\mathcal{M}}|^2 = \frac{1}{2}\sum_{\{s\}} |\mathcal{M}|^2 = \frac{e^4}{4p^4 \sin^4(\theta/2)} \left(E^2 - p^2 \sin^2(\theta/2)\right) ,$$

meaning that the cross section is given by

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha^2}{4|\vec{p}|^2 \beta^2 \sin^4 \frac{\theta}{2}} \left(1 - \beta^2 \sin^2 \frac{\theta}{2}\right) .$$

Problem 6 (20 points)

Consider a theory given by the following Lagrangian in 3 space-time dimensions ($c = \hbar = 1$)

$$\mathcal{L} = (\partial_\mu \varphi^*)(\partial^\mu \varphi) - m^2 \varphi^* \varphi - \frac{\lambda^2}{2}(\varphi^* \varphi)^2 - \kappa^3 (\varphi^* \varphi)^3 ,$$

where φ is a complex scalar field, and $m > 0$, $\lambda > 0$, $\kappa > 0$ are constants.

a) What are the mass dimensions of the following entities?

i) φ
ii) m
iii) λ
iv) κ

Solution [4 P]

i) $[\varphi] = M^{1/2}$
ii) $[m] = M$
iii) $[\lambda] = M^{1/2}$
iv) $[\kappa] = M^0$

b) Find the equations of motion.

Solution [4 P]

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} - \frac{\partial \mathcal{L}}{\partial \varphi} = 0 \quad \Leftrightarrow \quad \boxed{\partial_\mu^2 \varphi^* + m^2 \varphi^* + \lambda^2 (\varphi^* \varphi)\varphi^* + 3\kappa^3 (\varphi^* \varphi)^2 \varphi^* = 0}$$

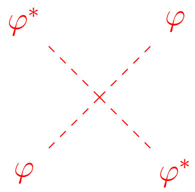
$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi^*)} - \frac{\partial \mathcal{L}}{\partial \varphi^*} = 0 \quad \Leftrightarrow \quad \boxed{\partial_\mu^2 \varphi + m^2 \varphi + \lambda^2 (\varphi^* \varphi)\varphi + 3\kappa^3 (\varphi^* \varphi)^2 \varphi = 0}$$

c) State the Feynman rules for all of the propagators and vertices of this theory. No derivation is necessary.

Solution [5 P]

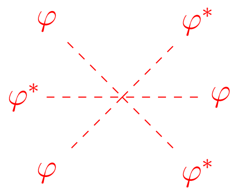
The required Feynman rules in momentum space:

Vertex λ:



$= -2i\lambda^2$

Vertex κ:



$= -(3!)^2 i\kappa^3$

Scalar propagator:

$$- - - - - = \frac{i}{p^2 - m^2 + i\varepsilon}$$

Consider the process

$$\varphi_{p_1}\varphi^*_{p_2} \to \varphi_{q_1}\varphi^*_{q_2}\varphi_{q_3}\varphi^*_{q_4} \ ,$$

where p_i $(i = 1, 2)$, and q_j $(j = 1, 2, 3, 4)$ are the incoming and outgoing momenta, respectively. We define the following variables

$$\begin{aligned} s &= (p_1 + p_2)^2 \ , \\ t_j &= (p_1 + p_2 - q_j)^2 \ , \\ u_j &= (p_1 - q_j - q_{j+1})^2 \ , \end{aligned}$$

for $j = 1, 2, 3, 4$ and $q_5 \equiv q_1$.

d) Show that

$$\sum_{j=1}^{4} t_j - 2s = 4m^2 \, .$$

Solution [5 P]

$$\begin{aligned} \sum_{j=1}^{4} t_j &= (p_1 + p_2 - q_1)^2 + (p_1 + p_2 - q_2)^2 + (p_1 + p_2 - q_3)^2 + (p_1 + p_2 - q_4)^2 \\ &= 4(p_1 + p_2)^2 - 2(p_1 + p_2) \cdot (q_1 + q_2 + q_3 + q_4) + q_1^2 + q_2^2 + q_3^2 + q_4^2 \ . \end{aligned}$$

Using

$$p_i^2 = q_i^2 = m^2 \quad \forall i \ ,$$

as well as energy & momentum conservation

$$p_1 + p_2 = q_1 + q_2 + q_3 + q_4 \ ,$$

we find

$$\sum_{j=1}^{4} t_j = 2(p_1 + p_2)^2 + 4m^2 = 2s + 4m^2 \ ,$$

meaning that

$$\sum_{j=1}^{4} t_j - 2s = 4m^2 \ .$$

e) Draw and label a Feynman diagram contributing to this process at tree level.

Solution [2 P]

There are many, so any reasonable diagram is OK.