Ludwig-Maximilians-Universität München

Quantum Field Theory (Quantum Electrodynamics)

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Guidelines:

- The exam consists of 7 problems.
- The duration of the exam is 96 hours.
- Please write your name or matriculation number on every sheet that you hand in.
- Your answers should be comprehensible and readable.

GOOD LUCK!

Exercise 1	8 P
Exercise 2	15 P
Exercise 3	25 P
Exercise 4	15 P
Exercise 5	25 P
Exercise 6	15 P
Exercise 7	20 P

Total	123 P

Problem 1 (8 points)

Simplify the following expressions as much as possible without using any representation of the γ matrices. The Minkowski metric convention is $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$. You may use the definitions $P_{\substack{L\\R}} = \frac{1}{2}(1 \mp \gamma^5)$ as well as the identities and relations you encountered during the course.

a) $\text{tr}[\gamma^\mu \gamma_\nu P_L \gamma_\mu P_R \gamma^\nu]$

b) $\text{tr}[(\gamma^\mu)^\dagger \gamma_\sigma \gamma_\nu P_R \gamma_\rho \gamma^5 P_L \gamma^\nu \gamma_\mu]$

c) $\text{tr}[\eta_{\mu\nu} \gamma^\rho \gamma_\sigma]$

d) $\exp[i\frac{\pi}{2}\gamma^5]$

Problem 2 (15 points)

Consider the following Lagrangian capturing the dynamics of a real scalar field ϕ in 4 space-time dimensions (we use units $c = \hbar = 1$)

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - c\phi^6 ,$$

with c a constant. The signature of Minkowski metric is mostly minus, i.e. $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$. Derive the corresponding interaction-picture Hamiltonian in terms of creation and annihilation operators.

Hint: The interaction-picture creation and annihilation operators $\hat{\alpha}_k^\dagger$ and $\hat{\alpha}_k$ are related to $\hat{a}_k^\dagger$ and $\hat{a}_k$ as

$$\hat{\alpha}_k^\dagger = \hat{a}_k^\dagger e^{i\omega_k t} , \quad \hat{\alpha}_k = \hat{a}_k e^{-i\omega_k t} .$$

Problem 3 (25 points)

Consider a theory given by the following Lagrangian density in 4 space-time dimensions (we use units $c = \hbar = 1$)

$$\mathcal{L} = \mathcal{L}_\varphi + \mathcal{L}_\psi + \mathcal{L}_{B_\mu} + \mathcal{L}_{\text{int}} ,$$

with

$$\begin{aligned}
\mathcal{L}_\varphi &= \frac{1}{2}(\partial_\mu \varphi)(\partial^\mu \varphi) - \frac{1}{2}M^2\varphi^2 , \\
\mathcal{L}_\psi &= \bar{\psi}(i\partial\!\!\!/ - m)\psi , \\
\mathcal{L}_{B_\mu} &= -\frac{1}{4}G_{\mu\nu}G^{\mu\nu} + \frac{\mu^2}{2}B_\mu B^\mu , \\
\mathcal{L}_{\text{int}} &= -\lambda\varphi\bar{\psi}\psi - \frac{g}{2}\varphi B_\mu B^\mu ,
\end{aligned}$$

where φ is a massive real scalar field of mass M, ψ is a spinor of mass m with $\bar{\psi} \equiv \psi^\dagger\gamma^0$, B_μ is a massive vector field of mass μ, with its field strength-tensor given by $G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$, and λ and g are constants.

a) What are the mass dimensions of the following entities?

   i) φ
   ii) ψ
   iii) M
   iv) m
   v) B_μ
   vi) μ
   vii) λ
   viii) g

b) Find the equations of motion.

c) State the Feynman rules for all of the propagators and vertices of this theory. No derivation is necessary.

d) Consider the process $BB \to \psi\bar{\psi}$. Draw and label the Feynman diagram(s) contributing to this process to leading order in λ and g.

e) Under what conditions is the above process kinematically allowed?

f) How many physical polarizations does the massive vector boson B_μ have? Explain.

g) Derive the spin-averaged amplitude squared for the above process in terms of the Mandelstam variables.

   Hint: Take the polarization vectors $\varepsilon_\mu^{(i)}$ to be real and use the fact that the sum over the physical polarization states of a spin-1 particle of mass $\mu \neq 0$ is $\sum_i \varepsilon_\mu^{(i)} \varepsilon_\nu^{(i)} = -\eta_{\mu\nu} + \frac{k_\mu k_\nu}{\mu^2}$.

h) Calculate the differential cross section of the process. Use the following general expression for the differential cross section of a 2-to-2 scattering process $AB \to CD$ in the center-of-mass (CM) frame:

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{CM}} = \frac{1}{2E_A 2E_B |\vec{v}_A - \vec{v}_B|} \frac{|\vec{p}_C|}{(2\pi)^2 4E_{\mathrm{CM}}} |\mathcal{M}(p_A, p_B \to p_C, p_D)|^2 \ ,$$

   where E_A, E_B and E_{CM} are the energies of A, B and the total initial energy, respectively. Also, $\vec{v}_X = \frac{\vec{p}_X}{E_X}$ for $X = A, B$. Finally, $|\mathcal{M}(p_A, p_B \to p_C, p_D)|^2$ is the spin-averaged amplitude squared you found in g).

i) Calculate the total cross section $\sigma = \frac{1}{N} \int \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{CM}} \mathrm{d}\Omega$. What is the normalization factor N? Explain briefly.

Problem 4 (15 points)

Consider the following action of a massless scalar field in $d > 2$ space-time dimensions

$$S = \int \mathrm{d}^d x \left[\frac{1}{2} (\partial_\mu \varphi)(\partial^\mu \varphi) - \lambda \varphi^{\frac{2d}{d-2}} \right] ,$$

with λ a real constant. The signature of Minkowski metric is mostly minus, i.e. $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, \ldots)$.

a) What is the mass dimension of φ?

b) Find the equations of motion.

c) Consider the following transformation

$$\varphi(x) \to \varphi'(x) = \alpha^\Delta \varphi(\alpha x) \ .$$

Determine Δ such that the action be invariant under this. What do you observe?

d) Find the corresponding Noether current. Show that it is conserved on the equations of motion.

Problem 5 (25 points)

a) A Dirac spinor transforms under Lorentz transformation as $\psi(x) \mapsto \psi'(x') = S(\Lambda)\psi(x)$. Using the Lorentz invariance of the Dirac equation, show that

$$S(\Lambda) = \exp(-\frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}) \ .$$

Under infinitesimal Lorentz transformation the spacetime coordinates transform as

$$x^\mu \mapsto x'^\mu = \Lambda^\mu_\nu x^\nu = (\delta^\mu_\nu + \omega^\mu_\nu)\, x^\nu \ .$$

Recall that $\sigma^{\mu\nu}$ is defined by $\sigma^{\mu\nu} = \frac{i}{2} [\gamma^\mu, \gamma^\nu]$.

Hint 1: Note that $\omega^{\mu\nu}$ is anti-symmetric.
Hint 2: You will need to prove $\left[\gamma^\lambda, \sigma^{\mu\nu}\right] = 2i \left[\eta^{\lambda\mu}\gamma^\nu - \eta^{\lambda\nu}\gamma^\mu\right]$.

b) Since the Dirac Lagrangian

$$\mathcal{L} = \bar{\psi} \left(i\gamma^\mu \partial_\mu - m\right) \psi \ ,$$

is Lorentz invariant, there is a corresponding conserved Noether current. Using Noether's theorem show that the current can be written as

$$J^\lambda = A_{\mu\nu} J^{\lambda,\mu\nu} \ ,$$

where

$$J^{\lambda,\mu\nu} = i\bar{\psi}\gamma^\lambda \left(-\frac{i}{4}\sigma^{\mu\nu} + x^\mu \partial^\nu \right) \psi \ ,$$

and $A_{\mu\nu}$ is an arbitrary anti-symmetric tensor.

c) Show that the current is conserved on the equations of motion.

d) Show that

$$J_k^\lambda = \varepsilon_{kij} J^{\lambda,ij} = \bar{\psi}\gamma^\lambda \left(S_k + L_k\right)\psi \ ,$$

where

$$S_k = \frac{\sigma_k}{2} \ , \qquad\qquad L_k = -i\varepsilon_{kij} x^i \partial_j \ .$$

Hint: Use the Dirac representation of the γ-matrices.

e) The corresponding charge is given by

$$Q_k = \int \mathrm{d}^3x \ \psi^\dagger(\vec{x}) \left(S_k + L_k\right)\psi(\vec{x}) \ .$$

Show that it satisfies the following equation

$$[Q_i, Q_j] = i\varepsilon_{ijk} Q_k \ .$$

f) Instead of a Dirac field, consider now a real scalar field ϕ with Lagrangian density

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2 \ .$$

Show that the current J_k^λ for ϕ is

$$J_k^\lambda = -i\left(\partial^\lambda\phi\right) L_k \phi \ .$$

g) Interpret the above results. Compare the current associated with the spinor field to the corresponding scalar field current found in the previous point.

Problem 6 (15 points)

Consider the theory

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi + \bar{\Psi}\left(i\gamma^\mu\partial_\mu - M\right)\Psi + g\bar{\psi}\gamma_\mu\Psi A^\mu \ .$$

Calculate the decay rate for the process $\Psi \to \psi\gamma$.
What are the conditions for this process to be kinematically allowed?

Problem 7 (20 points)

Consider the following action

$$S = \frac{1}{2}\int \mathrm{d}^4x \left[(\partial_\mu\varphi_1)(\partial^\mu\varphi_1) + (\partial_\mu\varphi_2)(\partial^\mu\varphi_2) + \frac{2\varphi_2}{\lambda}(\partial_\mu\varphi_1)(\partial^\mu\varphi_2) + \frac{\varphi_2^2}{\lambda^2}(\partial_\mu\varphi_2)(\partial^\mu\varphi_2)\right] \ ,$$

where φ_1 and φ_2 are two real massless scalar fields.

a) What is the mass dimension of λ?

b) Write down the Feynman rules.

c) Calculate the lowest order (in $1/\lambda^2$) Feynman amplitude for the process $\varphi_1\varphi_1 \to \varphi_2\varphi_2$.

Hint: You have to consider two Feynman diagrams.

d) Use the redefinition

$$\chi_1 = \varphi_1 + \frac{\varphi_2^2}{2\lambda}, \qquad \chi_2 = \varphi_2 \ ,$$

to rewrite the action.
What do you observe? With this new insight, interpret the result of point c).