1. INTERACTION PICTURE

Schrödinger picture: operators are time-independent
states are time-dependent

Interaction picture: operators are time-dependent
states — // —

$$O_I(t) = U_0^+(t)\, O_S\, U_0(t) \quad (1), \qquad |A,t\rangle_I = U_0^+(t)\,|A,t\rangle_S \quad (2),$$

$$U_0(t) = e^{-iH_0(t-t_0)} \quad (3)$$

① Trivial to show since H_0 commutes with itself.

② From (1), we have

$$H_{int}^I(t) = U_0^+(t)\, H_{int}^S\, U_0(t) = e^{iH_0(t-t_0)} H_{int}^S e^{-iH_0(t-t_0)}. \quad (4)$$

③ eom for operators:

$$\frac{d}{dt} O_I(t) = \frac{d}{dt}\left(U_0^+(t) O_S U_0(t)\right)$$

$$= \frac{d}{dt} U_0^+(t)\, O_S\, U_0(t) + U_0^+(t)\, O_S \frac{d}{dt} U_0(t). \quad (5)$$

since (follows from the definition eq. 3)

$$\frac{d}{dt} U_0^+(t) = \frac{d}{dt} e^{+iH_0(t-t_0)} = iH_0 U_0^+(t), \quad (6)$$

and $\frac{d}{dt} U_0(t) = \frac{d}{dt} e^{-iH_0(t-t_0)} = -iH_0 U_0(t)$, (7)

we find that (5) gives

$$\frac{d}{dt} O_I(t) = iH_0 U_0^+(t) O_S U_0(t) - i\, U_0^+(t) O_S U_0(t) H_0$$

$$= -i\left(O_I(t) H_0 - H_0 O_I(t)\right)$$

$$= -i\,[O_I(t), H_0]$$

$$\boxed{\rightarrow\ i\frac{d}{dt} O_I(t) = [O_I(t), H_0]\ .} \quad (8)$$

eom for states:

- $$\frac{d}{dt}|A,t\rangle_I = \frac{d}{dt}\left(U_0^+(t)|A,t\rangle_S\right)$$

$$= \frac{d}{dt}U_0^+(t)\,|A,t\rangle_S + U_0^+(t)\frac{d}{dt}|A,t\rangle_S$$

$$\overset{(6)}{=} iH_0 U_0^+(t)|A,t\rangle_S + U_0^+(t)\frac{d}{dt}|A,t\rangle_S$$

$$\Rightarrow \frac{d}{dt}|A,t\rangle_I = iH_0|A,t\rangle_I + U_0^+(t)\frac{d}{dt}|A,t\rangle_S. \quad (9)$$

- we know need to figure out the 2^{nd} term in (9).

Remember Schrödinger eq:

$$i\frac{d}{dt}|A,t\rangle_S = H^S|A,t\rangle_S = H_0^S|A,t\rangle_S + H_{int}^S|A,t\rangle_S$$

Now use that $H_0^S = H_0$, to get

$$i\frac{d}{dt}|A,t\rangle_S = H_0|A,t\rangle_S + H_{int}^S|A,t\rangle_S \quad ,$$

therefore

$$\frac{d}{dt}|A,t\rangle_S = -iH_0|A,t\rangle_S - iH_{int}^S|A,t\rangle_S \quad . (10)$$

- Plug (10) into (9), to obtain:

$$\frac{d}{dt}|A,t\rangle_I = iH_0|A,t\rangle_I - iU_0^+(t)H_0|A,t\rangle_S - iU_0^+(t)H_{int}^S|A,t\rangle_S$$

$$= iH_0|A,t\rangle_I - iH_0U_0^+(t)|A,t\rangle_S - iU_0^+(t)H_{int}^S|A,t\rangle_S$$

$$\frac{d}{dt}|A,t\rangle_I \overset{(2)}{=} \cancel{iH_0|A,t\rangle_I - iH_0|A,t\rangle_I} - iU_0^+(t)H_{int}^S|A,t\rangle_S. \quad (11)$$

From (2) we find

$$|A,t\rangle_S = U_0(t)|A,t\rangle_I \quad , (12)$$

- meaning that (11) gives us

$$\frac{d}{dt}|A,t\rangle_I = -iU_0^+(t)H_{int}^S U_0(t)|A,t\rangle_I = -iH_{int}^I|A,t\rangle_I$$

$$\rightarrow \boxed{i\frac{d}{dt}|A,t\rangle_I = H_{int}^I|A,t\rangle_I.} \quad (13)$$

④ **Heisenberg picture**: operators are time-dependent
states are time-independent

~~Start from~~ we know that

$$O_H(t) = e^{iH(t-t_0)} O_S\, e^{-iH(t-t_0)} , \quad (14)$$

$$|A\rangle_H = e^{iH(t-t_0)} |A,t\rangle_S \quad . (15)$$

Start from eq. (1)

$$O_I(t) = U_0^+(t)\, O_S\, U_0(t) = U_0^+(t)\, e^{-iH(t-t_0)} \underbrace{e^{iH(t-t_0)} O_S\, e^{-iH(t-t_0)}} e^{iH(t-t_0)} U_0(t)$$

$$\overset{(14)}{=} U_0^+(t)\, e^{-iH(t-t_0)} O_H(t)\, e^{iH(t-t_0)} U_0(t)$$

$$\rightarrow \boxed{O_I(t) = U(t,t_0)\, O_H(t)\, U^+(t,t_0)} \quad (16)$$

with

$$U(t,t_0) = U_0^+(t)\, e^{-iH(t-t_0)} = e^{iH_0(t-t_0)} e^{-iH(t-t_0)} \quad . (17)$$

Now start from (2)

$$|A,t\rangle_I = U_0^+(t)\, |A,t\rangle_S = U_0^+(t)\, e^{-iH(t-t_0)} e^{iH(t-t_0)} |A,t\rangle_S$$

$$\overset{(5)}{=} U_0^+(t)\, e^{-iH(t-t_0)} |A\rangle_H$$

$$\rightarrow \boxed{|A,t\rangle_I \overset{(17)}{=} U(t,t_0)|A\rangle_H} \quad (18)$$

⑤ Differentiate $U(t,t_0)$ wrt time:

$$i\frac{\partial}{\partial t} U(t,t_0) = i\frac{\partial}{\partial t}\left(e^{iH_0(t-t_0)} e^{-iH(t-t_0)}\right)$$

$$= i\left(iH_0\, e^{iH_0(t-t_0)} e^{-iH(t-t_0)} - i\, e^{iH_0(t-t_0)} H\, e^{-iH(t-t_0)}\right)$$

$$= -\left(H_0\, e^{iH_0(t-t_0)} e^{-iH(t-t_0)} - e^{iH_0(t-t_0)} (H_0 + H_{int}^S) \times e^{-iH(t-t_0)}\right)$$

$$= + e^{iH_0(t-t_0)} H_{int}^S\, e^{-iH(t-t_0)}$$

$$= e^{iH_0(t-t_0)} H_{int}^S\, e^{-iH_0(t-t_0)} e^{iH_0(t-t_0)} e^{-iH(t-t_0)}$$

$$\boxed{i\frac{\partial}{\partial t} U(t,t_0) = H_{int}^I(t)\, e^{iH_0(t-t_0)} e^{-iH(t-t_0)} = H_{int}^I\, U(t,t_0)} \quad (19)$$

Now check what happens for $t=t_0$

$$i\frac{\partial}{\partial t}U(t,t_0)\Big|_{t=t_0} = H^I_{int}(t_0)\,U(t_0,t_0)$$

$$\rightarrow i\left(iH_0 e^{iH_0(t-t_0)}e^{-iH(t-t_0)} - i e^{iH_0(t-t_0)} H e^{-iH(t-t_0)}\right)_{t=t_0}$$

$$= H^I_{int}(t_0)U(t_0,t_0)$$

$$\rightarrow i(iH_0 - iH) = H^I_{int}(t_0)U(t_0,t_0)$$

$$\rightarrow i(iH_0 - iH_0 - iH_{int}) = H^I_{int}(t_0)\,U(t_0,t_0)$$

$$\rightarrow H_{int} = H^I_{int}(t_0)\,U(t_0,t_0)$$

since

$$H^I_{int}(t) = e^{iH_0(t-t_0)} H^S_{int} e^{-iH_0(t-t_0)} = e^{iH_0(t-t_0)} H_{int} e^{-iH_0(t-t_0)}$$

$$\rightarrow H^I_{int}(t_0) = H_{int} \quad ,$$

we get

$$H_{int} = H_{int}\,U(t_0,t_0)$$

meaning that

$$U(t_0,t_0) = 1 \quad . \quad (20)$$

⑥ We have

$$U(t,t_0) = Te^{-i\int_{t_0}^{t}dt'\,H^I_{int}(t')} \quad . \quad (21)$$

Then,

$$i\frac{\partial}{\partial t}\left[Te^{-i\int_{t_0}^{t}dt'\,H^I_{int}(t')}\right] = T H^I_{int}(t)e^{-i\int_{t_0}^{t}dt'\,H^I_{int}(t')}$$

$$= H^I_{int}\,Te^{-i\int_{t_0}^{t}dt'\,H^I_{int}(t')}$$

$$\rightarrow \boxed{i\frac{\partial}{\partial t}U(t,t_0) = H^I_{int}\,U(t,t_0)} \quad . \quad (22)$$

Time ordering is essential, because otherwise you would need to impose

$$[H^I_{int}(t), H^I_{int}(t')]_{t\neq t'} = 0 \quad , \quad (23)$$

which may not be the case. (show that!)

⊗ generalization to arbitrary time:

$$U(t,t') = e^{iH_0(t-t_0)} e^{-iH(t-t')} e^{-iH_0(t'-t_0)} \quad , \quad (24)$$

such that

$$U(t',t') = 1 \quad . \quad (25)$$

For $t_1 \geq t_2 \geq t_3$, we have

$$U(t_1,t_2)\, U(t_2,t_3) = e^{iH_0(t_1-t_0)} e^{-iH(t_1-t_2)} \cancel{e^{-iH_0(t_2-t_0)}} \cdot$$
$$\cdot \cancel{e^{iH_0(t_2-t_0)}} e^{-iH(t_2-t_3)} e^{-iH_0(t_3-t_0)}$$
$$= e^{iH_0(t_1-t_0)} e^{-iH(t_1-t_2)} e^{-iH(t_2-t_3)} e^{-iH_0(t_3-t_0)}$$
$$= e^{iH_0(t_1-t_0)} e^{-iH(t_1-t_3)} e^{-iH_0(t_3-t_0)}$$

$$\boxed{\rightarrow U(t_1,t_2)U(t_2,t_3) = U(t_1,t_3)} \quad . \quad (26)$$

Similarly,

$$\boxed{U(t_1,t_3)\, U^\dagger(t_2,t_3) = U(t_1,t_2)} \quad (27)$$

⊗ unitarity:

$$U(t,t')\, U^\dagger(t,t') = e^{iH_0(t-t_0)} e^{-iH(t-t')} \cancel{e^{-iH_0(t'-t_0)}} \cdot$$
$$\cdot \cancel{e^{iH_0(t'-t_0)}} e^{iH(t-t')} e^{-iH_0(t-t_0)}$$
$$= e^{iH_0(t-t_0)} \cancel{e^{-iH(t-t')}} \cancel{e^{iH(t-t')}} e^{-iH_0(t-t_0)}$$
$$= \cancel{e^{iH_0(t-t_0)}} \cancel{e^{-iH_0(t-t_0)}} = 1$$

$$\rightarrow \boxed{U(t,t')\, U^\dagger(t,t') = 1} \quad (28) .$$

3. Box normalization

The replacements can be found from the well-know particle in a box. The momenta are

$$(p_x, p_y, p_z) = \frac{2\pi}{L}(n_x, n_y, n_z) \, ,$$

meaning that

$$d^3\vec{p} = \frac{(2\pi)^3}{L^3} d^3\vec{n} = \frac{(2\pi)^3}{V} dN \rightarrow dN = \frac{V d^3\vec{p}}{(2\pi)^3} .$$

① We know that

$$[a_p, a^\dagger_{p'}] = (2\pi)^3 2\omega_p \delta^{(3)}(\vec{p} - \vec{p}') \quad (1)$$

$$= (2\pi)^3 \sqrt{2\omega_p}\sqrt{2\omega_{p'}}\, \delta^{(3)}(\vec{p} - \vec{p}')$$

since

$$(2\pi)^3 \delta^{(3)}(\vec{p} - \vec{p}') \rightarrow V\delta_{p,p'} \, , \quad (2)$$

we find from (1)

$$[a_p, a^\dagger_{p'}] = \sqrt{2V\omega_p}\sqrt{2V\omega_{p'}}\, \delta_{p,p'} \, . \quad (3)$$

Thus,

$$\left[\frac{a_p}{\sqrt{2V\omega_p}}, \frac{a^\dagger_{p'}}{\sqrt{2V\omega_{p'}}}\right] = \delta_{p,p'} \, , \quad (4)$$

which upon introducing

$$(\tilde{a}_p, \tilde{a}^\dagger_{p'}) = \left(\frac{a_p}{\sqrt{2V\omega_p}}, \frac{a^\dagger_{p'}}{\sqrt{2V\omega_{p'}}}\right) \, , \quad (5)$$

the above becomes

$$[\tilde{a}_p, \tilde{a}^\dagger_{p'}] = \delta_{p,p'} \, . \quad (6)$$

Similarly, we find for the $b_p, b^\dagger_{p'}$ operators that

$$[\tilde{b}_p, \tilde{b}^\dagger_{p'}] = \delta_{p,p'} \, , \text{ with } (\tilde{b}_p, \tilde{b}^\dagger_{p'}) = \left(\frac{b_p}{\sqrt{2V\omega_p}}, \frac{b^\dagger_{p'}}{\sqrt{2V\omega_{p'}}}\right) . \quad (7)$$

② From

$$\chi = \int \frac{d^3\vec{p}}{(2\pi)^3 2\omega_p} \left(a_p e^{-ipx} + b_p^+ e^{ipx}\right) \qquad ,(8)$$

we find

$$\chi = \sum_p \frac{1}{\sqrt{2V\omega_p}} \left(\tilde{a}_p e^{-ipx} + \tilde{b}_p^+ e^{ipx}\right) \qquad . (9)$$

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2. DISCRETE SYMMETRIES

ⓐ Parity

① We have that

$$\varphi(t,\vec{x}) \to \tilde{\varphi}(t,\vec{x}) = P\varphi(\cancel{t},\cancel{\vec{x}})P^{-1} = \eta_P \varphi(t,-\vec{x}) \quad (1)$$

Let's expand the Lagrangian as

$$\tilde{\mathcal{L}} = \frac{1}{2}\left\{ \left(\frac{\partial\tilde{\varphi}(t,\vec{x})}{\partial t}\right)^2 - \left(\frac{\partial\tilde{\varphi}(t,\vec{x})}{\partial x_i}\right)^2 - m^2\tilde{\varphi}^2(t,\vec{x})\right\}$$

$$\overset{(1)}{=} \frac{1}{2}\left\{ \left(\frac{\partial\varphi(t,-\vec{x})}{\partial t}\right)^2 - \left(\frac{\partial\varphi(t,-\vec{x})}{\partial x_i}\right)^2 - m^2\varphi^2(t,-\vec{x})\right\}$$

$$= \frac{1}{2}\left\{ \left(\frac{\partial\varphi(t,-\vec{x})}{\partial t}\right)^2 - \left(\frac{\partial\varphi(t,-\vec{x})}{\partial(-x_i)}\right)^2 - m^2\varphi^2(t,-\vec{x})\right\} , \quad (2)$$

since $\eta_P^2 = 1$

Then

$$\tilde{S} = \int d^4x\, \tilde{\mathcal{L}} = \frac{1}{2}\int dt\, d^3\vec{x} \left\{ \left(\frac{\partial\varphi(t,-\vec{x})}{\partial t}\right)^2 - \left(\frac{\partial\varphi(t,-\vec{x})}{\partial(-x_i)}\right)^2 - m^2\varphi^2(t,-\vec{x})\right\}$$

$$= \frac{1}{2}\int dt\, d^3\vec{y} \left\{ \left(\frac{\partial\varphi(t,\vec{y})}{\partial t}\right)^2 - \left(\frac{\partial\varphi(t,\vec{y})}{\partial y_i}\right)^2 - m^2\varphi^2(t,\vec{y})\right\}$$

$$= \frac{1}{2}\int d^4x \left((\partial_\mu\varphi)^2 - m^2\varphi^2\right) = S \quad , \quad (3)$$

as it should.

② Start from the following

$$|\vec{p}_1,\dots,\vec{p}_n\rangle = a^+_{p_1}\dots a^+_{p_n}|0\rangle \quad . \qquad (4)$$

Then

$$P|\vec{p}_1\dots\vec{p}_n\rangle = P a^+_{p_1}\dots a^+_{p_n}|0\rangle$$
$$= P a^+_{p_1}P^{-1}P a^+_{p_2}P^{-1}\dots P a^+_{p_n}P^{-1}P|0\rangle$$
$$= P a^+_{p_1}P^{-1}\dots P a^+_{p_n}P^{-1}|0\rangle \quad , \qquad (5)$$

since $P|0\rangle = |0\rangle$.

Thus, we need to find how the ladder operators behave under parity. We consider

$$P\varphi(t,\vec{x})P^{-1} = n_P\,\varphi(t,-\vec{x}) \quad , \qquad (6)$$

meaning that

$$\sum_p \frac{1}{\sqrt{2V\omega_p}}\left(P a_p P^{-1} e^{-i\omega_p t + i\vec{p}\vec{x}} + P a_p^+ P^{-1} e^{-i\omega_p t - i\vec{p}\vec{x}}\right)$$
$$\overset{(6)}{=} n_P\sum_p \frac{1}{\sqrt{2V\omega_p}}\left(a_p e^{-i\omega_p t - i\vec{p}\vec{x}} + a_p^+ e^{-i\omega_p t + i\vec{p}\vec{x}}\right) . \qquad (7)$$

Therefore,

$$P a_p P^{-1} = n_p a_{-p} \quad , \quad P a_p^+ P^{-1} = n_p a^+_{-p} \qquad (8)$$

Plugging (8) into (5), we obtain

$$P|\vec{p}_1\dots\vec{p}_n\rangle = n_p a^+_{-p_1} n_p a^+_{-p_2}\dots n_p a^+_{-p_n}|0\rangle$$
$$= (n_p)^n a^+_{-p_1} a^+_{-p_2}\dots a^+_{-p_n}|0\rangle$$
$$= (n_p)^n |-\vec{p}_1\dots-\vec{p}_n\rangle \quad . \qquad (9)$$

③ We know that for 2 operators A, B, the following holds

$$e^{i\alpha A}Be^{-i\alpha A} = \sum_{n=0}^{+\infty}\frac{(i\alpha)^n}{n!}B_n \quad . \qquad (10)$$

In our case,

$$B_0 = B = a_p \quad , \quad B_n = [A, B_{n-1}] \, , \; n = 1, 2, \dots \qquad (11)$$

$$\alpha = -\frac{\pi}{2} \, , \quad A = \sum_k a_k^+ a_k$$

Then, we get

$\circledast$ $B_1 = [A, B_0] = \sum_k [a_k^+ a_k, a_p]$

$$= \sum_k \left(a_k^+ \cancel{[a_k, a_p]}^{0} + [a_k^+, a_p] a_k \right)$$

$$= \sum_k \left(-[a_p, a_k^+] a_k \right) = - \sum_k \delta_{p,k} a_k = -a_p$$

$\circledast$ $B_2 = [A, B_1] = - \sum_k [a_k^+ a_k, a_p] = - \sum_k \left(a_k^+ [a_k, a_p] + [a_k^+, a_p] a_k \right)$

$$= \sum_k [a_p, a_k^+] a_k = a_p$$

The pattern is already clear:

$$B_n = (-1)^n a_p \qquad (12)$$

Therefore

$$\mathcal{L}_1 a_p \mathcal{L}_1^{-1} = \sum_{n=0}^{+\infty} \left(-\frac{i\pi}{2}\right)^n \frac{1}{n!} (-1)^n a_p = \sum_{n=0}^{+\infty} \left(\frac{i\pi}{2}\right)^n \frac{1}{n!} a_p$$

$$\Rightarrow \mathcal{L}_1 a_p \mathcal{L}_1^{-1} = e^{\frac{i\pi}{2}} a_p = i a_p \quad . \qquad (13)$$

For $\mathcal{L}_2$, we work in similar manner. Now, we take

$$\alpha = \frac{\pi}{2} n_\ell , \quad B_0 = a_p , \quad B_n = [A, B_{n-1}] , \quad n = 1, 2, \ldots , \quad A = \sum_k a_k^+ a_{-k} \qquad (14)$$

Then we find

$\circledast$ $B_1 = [A, B_0] = -a_{-p}$

$\circledast$ $B_2 = [A, B_1] = +a_p$

$\vdots$

The pattern is also clear here:

$$B_n = (-1)^n a_{(-1)^n p} \quad . \qquad (15)$$

Then,

$$\mathcal{L}_2 a_p \mathcal{L}_2^{-1} = \sum_{n=0}^{+\infty} \left(\frac{i\pi}{2} n_p\right)^n \frac{1}{n!} (-1)^n a_{(-1)^n p} . \qquad (16)$$

The above form invites us to split the sum into even & odd parts. The even sum to $ch\left(\frac{i\pi}{2} n_p\right) = 0$ for $n_p = \pm 1$.

we are left only with the $\sin(\frac{i\pi}{2}\eta_p)$ part, that gives

$$\mathcal{P}_2 a_p \mathcal{P}_2^{-1} = -i\eta_p a_{-p} \quad . \quad (17)$$

④ $$\mathcal{P}^+\mathcal{P} = (\mathcal{P}_1\mathcal{P}_2)^+\mathcal{P}_1\mathcal{P}_2 = \mathcal{P}_2^+\cancel{\mathcal{P}_1^+\mathcal{P}_1}^{1}\mathcal{P}_2 = \cancel{\mathcal{P}_2^+\mathcal{P}_2}^{1} = 1 \quad (18)$$

→ unitary operator

⑤ In the continuum normalization,

$$\mathcal{P}_1 = e^{-\frac{i\pi}{2}\int\frac{d^3\vec{p}}{(2\pi)^3 2\omega_p} a_p^+ a_p}, \quad \mathcal{P}_2 = e^{\frac{i\pi}{2}\eta_p\int\frac{d^3\vec{p}}{(2\pi)^3 2\omega_p} a_p^+ a_{-p}} \quad . \quad (19)$$

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⑥ charge conjugation

we are given that

$$\chi \to \tilde{\chi} = \mathcal{C}\chi\mathcal{C}^{-1} = \eta_c \chi^+ \quad . \quad (20)$$

Then

$$\chi^+ \to \tilde{\chi}^+ = \eta_c^* \chi \quad . \quad (21)$$

The normal-ordered Lagrangian becomes

$$\tilde{\mathcal{L}} = :\partial_\mu\tilde{\chi}^+\partial^\mu\tilde{\chi} - m^2\tilde{\chi}^+\tilde{\chi}:$$

$$= :\partial_\mu\chi\,\partial^\mu\chi^+ - m^2\chi\chi^+:$$

$$= :\partial_\mu\chi^+\partial^\mu\chi - m^2\chi^+\chi: = \mathcal{L} \quad . \quad (22)$$

The $U(1)$ Noether current is

$$J_\mu = i(\partial_\mu\chi^+\chi - \partial_\mu\chi\,\chi^+) \quad , \quad (23)$$

meaning that

$$\tilde{J}_\mu = i(\partial_\mu\chi\,\chi^+ - \partial_\mu\chi^+\chi) = -J_\mu \quad , \quad (24)$$

i.e. flips sign under charge conjugation.

(2) We know that

$$\mathcal{C}\chi\mathcal{C}^{-1} = \eta_c \chi^+$$

$$\to \sum_p \frac{1}{\sqrt{2V\omega_p}} \left(\mathcal{C}a_p\mathcal{C}^{-1}e^{-ipx} + \mathcal{C}b_p^+\mathcal{C}^{-1}e^{ipx}\right)$$
$$= \eta_c \sum_p \frac{1}{\sqrt{2V\omega_p}} \left(b_p e^{-ipx} + a_p^+ e^{ipx}\right), \quad (25)$$

which tranlates into

$$\mathcal{C}a_p\mathcal{C}^{-1} = \eta_c b_p, \quad \mathcal{C}b_p\mathcal{C}^{-1} = \eta_c^* a_p, \quad (26)$$

and

$$\mathcal{C}a_p^+\mathcal{C}^{-1} = \eta_c^* b_p^+, \quad \mathcal{C}b_p^+\mathcal{C}^{-1} = \eta_c a_p^+. \quad (27)$$

(3) $$\mathcal{C}|a,\vec{p}\rangle = \mathcal{C}a_p^+|0\rangle = \mathcal{C}a_p^+\mathcal{C}^{-1}\mathcal{C}|0\rangle = \mathcal{C}a_p^+\mathcal{C}^{-1}|0\rangle \overset{(27)}{=} \eta_c^* b_p^+|0\rangle = \eta_c^*|b,\vec{p}\rangle. \quad (28)$$

$$\mathcal{C}|b,\vec{p}\rangle = \mathcal{C}b_p^+\mathcal{C}^{-1}|0\rangle = \eta_c a_p^+|0\rangle = \eta_c|a,\vec{p}\rangle. \quad (29)$$

$\Rightarrow$ $\mathcal{C}$ operation exchanges particles ($a^+|0\rangle$) for the corresponding antiparticles ($b^+|0\rangle$).