Quantum Field Theory (Quantum Electrodynamics)

Problem Set 9 18 & 20 December 2023

1. Interaction Picture

Consider a theory with the following Hamiltonian in the Schrödinger Picture (S.P.)

$$H^S = H_0^S + H_{\text{int}}^S \ ,$$

where H_0^S is the free part and H_{int}^S contains interaction terms.

In the Interaction Picture (I.P.), both the operators and states are time-dependent and are related to the ones of the S.P. as follows

$$\mathcal{O}_I(t) = U_0^+(t)\mathcal{O}_S U_0(t) \qquad |A,t\rangle_I = U_0^+(t)\,|A,t\rangle_S \ ,$$

with

$$U_0(t) = e^{-iH_0(t-t_0)} \ ,$$

the evolution operator depending only on the free part of the Hamiltonian.

1. Show that the free part of the Hamiltonian in the I.P. satisfies
$$H_0^I = H_0^S \equiv H_0$$
2. Find how the interacting part of the Hamiltonian in the I.P. H_{int}^I is related to H_{int}^S.
3. Find the I.P. equations of motion for the operators and states.
4. Show that $\mathcal{O}_I(t)$ and $|A,t\rangle_I$ are related to the operators and states in the Heisenberg picture, $\mathcal{O}_H(t)$ and $|A\rangle_H$ respectively, as
$$\mathcal{O}_I(t) = U(t,t_0)O_H(t)U^+(t,t_0) \ , \quad |A,t\rangle_I = U(t,t_0)\,|A\rangle_H \ ,$$
where $U(t,t_0) = e^{iH_0(t-t_0)}e^{-iH(t-t_0)}$ is the time-evolution operator in the I.P.
5. Show that $U(t,t_0)$ is the unique solution to the following differential equation with a suitable initial condition (which one ?)
$$i\frac{\partial}{\partial t}U(t,t_0) = H_{\text{int}}^I(t)U(t,t_0) \ .$$
6. Show that the solution of this differential equation can be written as
$$\begin{aligned} U(t,t_0) &= T\exp\left[-i\int_{t_0}^{t} \mathrm{d}t' H_{\text{int}}^I(t')\right] \\ &\equiv 1 + (-i)\int_{t_0}^{t} \mathrm{d}t_1 H_{\text{int}}^I(t_1) + \frac{(-i)^2}{2!}\int_{t_0}^{t} \mathrm{d}t_1 \mathrm{d}t_2 T\{H_{\text{int}}^I(t_1)H_{\text{int}}^I(t_2)\} + \ldots . \end{aligned}$$
Why is time-ordering essential ? What is the appropriate generalization for a time t' other than the reference time t_0. Show that for $t_1 \geq t_2 \geq t_3$ we have
$$U(t_1,t_2)U(t_2,t_3) = U(t_1,t_3), \quad U(t_1,t_3)[U(t_2,t_3)]^+ = U(t_1,t_2).$$
Is $U(t_1,t_2)$ unitary ?

2. Discrete symmetries : Parity & Charge conjugation

a) The *parity* transformation of the Hermitian Klein-Gordon field $\phi(x)$ is defined as

$$\phi(t,\vec{x}) \to \mathcal{P}\phi(t,\vec{x})\mathcal{P}^{-1} = \eta_{\mathcal{P}}\phi(t,-\vec{x}) \ , \tag{1}$$

where the parity operator $\mathcal{P}$ is a unitary operator that leaves the vacuum invariant, $\mathcal{P}|0\rangle = |0\rangle$, and $\eta_{\mathcal{P}} = \pm 1$ is called the intrinsic parity of the field.

1. Show that the parity transformation leaves the Lagrangian of the free massive scalar field invariant.
2. Show that for an arbitrary n-particle state

$$\mathcal{P}|\vec{p}_1,\vec{p}_2,\ldots,\vec{p}_n\rangle = \eta_{\mathcal{P}}^n |-\vec{p}_1,-\vec{p}_2,\ldots,-\vec{p}_n\rangle \ .$$

3. Working with the box normalization, prove that

$$\mathcal{P}_1\hat{a}(\vec{p})\mathcal{P}_1^{-1} = i\hat{a}(\vec{p}) \ , \quad \mathcal{P}_2\hat{a}(\vec{p})\mathcal{P}_2^{-1} = -i\eta_P\hat{a}(-\vec{p}) \ ,$$

where $\hat{a}(\vec{p})$ are the anihhilation operators of the field, and $\mathcal{P}_1$ and $\mathcal{P}_2$ are given by

$$\mathcal{P}_1 = \exp\left(-i\frac{\pi}{2}\sum_{\vec{p}}\hat{a}^+(\vec{p})\hat{a}(\vec{p})\right) \ , \quad \mathcal{P}_2 = \exp\left(i\frac{\pi}{2}\eta_P\sum_{\vec{p}}\hat{a}^+(\vec{p})\hat{a}(-\vec{p})\right) \ .$$

4. Show that the operator $\mathcal{P} = \mathcal{P}_1\mathcal{P}_2$ is unitary and satisfies eq. (1).
5. Find the form of $\mathcal{P}_1$ and $\mathcal{P}_2$ in the continuum normalization.

b) *Charge conjugation* for the complex Klein-Gordon field $\chi(x)$ is defined by

$$\chi(x) \to \mathcal{C}\chi(x)\mathcal{C}^{-1} = \eta_{\mathcal{C}}\chi^+(x) \ , \tag{2}$$

where $\mathcal{C}$ is a unitary operator which leaves the vacuum invariant, $\mathcal{C}|0\rangle = |0\rangle$, and is called the charge conjugation operator ; $\eta_{\mathcal{C}}$ is a phase factor.

1. How does the (normal-ordered) Lagrangian density

$$\mathcal{L} =: |\partial_\mu\chi|^2 - m^2|\chi|^2 : \ ,$$

transform under charge conjugation ? What about the Noether current j_μ associated with the global $U(1)$ symmetry ?
2. Find how the ladder operators $\hat{a}(\vec{p})$, $\hat{b}(\vec{p})$ and their conjugates transform under (2).
3. Show that the charge conjugation operator interchanges particles and antiparticles, i.e.

$$\mathcal{C}|a,\vec{p}\rangle = \eta_{\mathcal{C}}^*|b,\vec{p}\rangle \ , \quad \mathcal{C}|b,\vec{p}\rangle = \eta_{\mathcal{C}}|a,\vec{p}\rangle \ ,$$

where $|i,\vec{p}\rangle$ denotes the state with a single i-particle of momentum $\vec{p}$.

3. Box normalization

Like in statistical mechanics, it is sometimes useful to put a field theory inside a box. An infinitesimal volume $\mathrm{d}^3\vec{p}$ in momentum space contains $V\mathrm{d}^3\vec{p}/(2\pi)^3$ states, therefore when passing from the continuum to the box normalization we have to make the following replacement

$$\int \frac{d^3\vec{p}}{(2\pi)^3} f(\vec{p}) \leftrightarrow \frac{1}{V}\sum_{\vec{p}} f(\vec{p}) \ ,$$

where V is the volume of the box. The above relation implies that the correspondence between the delta function and Kronecker delta reads

$$\delta^{(3)}(\vec{p}-\vec{q}) \leftrightarrow \frac{V}{(2\pi)^3}\delta_{\vec{p},\vec{q}} \ .$$

Let us start with the familiar expression for the complex scalar field

$$\chi(x) = \int \frac{d^3\vec{p}}{(2\pi)^3 2\omega_{\vec{p}}} \left(\hat{a}_{\vec{p}} e^{-ipx} + \hat{b}^+_{\vec{p}} e^{ipx}\right) ,$$

with the shorthand notation $\hat{a}_{\vec{p}} = \hat{a}(\vec{p})$ and similarly $\hat{b}_{\vec{p}} = \hat{b}(\vec{p})$.

1. Redefine appropriately the operators $\hat{a}_{\vec{p}}$ and $\hat{a}^+_{\vec{p}}$, such that their commutation relation becomes $[\hat{a}_{\vec{p}}, \hat{a}^+_{\vec{q}}] = \delta_{\vec{p},\vec{q}}$. Do the same for $\hat{b}_{\vec{p}}$ and $\hat{b}^+_{\vec{p}}$. With this normalization, the state $|\vec{p}\rangle \equiv \hat{a}^+_{\vec{p}}|0\rangle$ is normalized to one.
2. Show that in terms of the new ladder operators the scalar field decomposition reads

$$\chi(x) = \sum_{\vec{p}} \frac{1}{\sqrt{2V\omega_{\vec{p}}}} \left(\hat{a}_{\vec{p}} e^{-ipx} + \hat{b}^+_{\vec{p}} e^{ipx}\right) .$$