Quantum Field Theory (Quantum Electrodynamics)

Problem Set 8 11 & 13 December 2023

1. Propagators

1. We have seen that the real scalar field admits the following mode expansion

$$\hat{\phi}(x) = \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3 2\omega_{\vec{p}}} \left[\hat{a}(\vec{p})e^{-ip\cdot x} + \hat{a}(\vec{p})^+ e^{ip\cdot x}\right] \ , \tag{1}$$

with $p \cdot x = \omega_{\vec{p}} t - \vec{p}\cdot\vec{x}$.

(a) Compute the Wightman function

$$\mathcal{D}(x - x') = \langle 0| \hat{\phi}(x)\hat{\phi}(x') |0\rangle \ . \tag{2}$$

(b) Starting from

$$\mathcal{D}_F(x, x') = \langle 0| T\hat{\phi}(x)\hat{\phi}(x') |0\rangle \ , \tag{3}$$

with T the time-ordered product

$$T\hat{\phi}(x)\hat{\phi}(x') = \theta(t - t')\hat{\phi}(x)\hat{\phi}(x') + \theta(t' - t)\hat{\phi}(x')\hat{\phi}(x) \ , \tag{4}$$

show that

$$\mathcal{D}_F(x, x') = \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3 2\omega_{\vec{p}}} \left(e^{-ip\cdot(x-x')}\theta(t - t') + e^{ip\cdot(x-x')}\theta(t' - t)\right) \ . \tag{5}$$

Prove that the above can be written as

$$\mathcal{D}_F(x, x') = \lim_{\epsilon\to 0} \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{ip\cdot(x-x')} \ . \tag{6}$$

Useful formula :

$$e^{-i\omega_{\vec{p}}\tau}\theta(\tau) + e^{i\omega_{\vec{p}}\tau}\theta(-\tau) = i \lim_{\epsilon\to 0} \frac{2\omega_{\vec{p}}}{2\pi} \int \mathrm{d}\omega \frac{e^{i\omega\tau}}{\omega^2 - \omega_{\vec{p}}^2 + i\epsilon} \ , \quad \epsilon > 0 \ .$$

(c) Check that for $\epsilon = 0$ the Feynman propagator satisfies the Klein-Gordon equation

$$(\Box_x + m^2)\mathcal{D}_F(x, x') = -i\delta^{(4)}(x - x') \ . \tag{7}$$

(d) Show that in position space the Feynman propagator for a massless scalar field is

$$\mathcal{D}_F(x, x') = -\frac{1}{4\pi^2} \lim_{\epsilon\to 0} \frac{1}{(x - x')^2 - i\epsilon} \ . \tag{8}$$

Hint : Start from eq. (9).
Useful formula :

$$\int_0^\infty \mathrm{d}\mu e^{i\mu a} = \lim_{\epsilon\to 0} \frac{i}{a + i\epsilon} \ .$$

2. Compute the Feynman propagator $\mathcal{S}_F(x,x')$ for the Dirac field.

Hint : You may use the fact that

$$\mathcal{S}_F(x,x') = (-i\gamma^\mu \frac{\partial}{\partial x^\mu} - m)\mathcal{D}_F(x,x') \ ,$$

with $\mathcal{D}_F(x,x')$ the Feynman propagator for the scalar field, eq. (10)

2. Contractions

When computing correlation functions, it is useful to decompose the fields into positive and negative frequency parts. For the (free) real scalar field, this amounts to writing it as

$$\hat{\phi}(x) = \hat{\phi}^+(x) + \hat{\phi}^-(x) \ ,$$

with

$$\hat{\phi}^+(x) = \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3 2\omega_{\vec{p}}} \hat{a}(\vec{p}) e^{-ipx} \ , \quad \hat{\phi}^-(x) = \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3 2\omega_{\vec{p}}} \hat{a}^+(\vec{p}) e^{ipx} \ ,$$

such that $\hat{\phi}^+(x)\left|0\right\rangle = 0$ and $\left\langle 0\right| \hat{\phi}^-(x) = 0$.

1. Show that

$$T\hat{\phi}(x)\hat{\phi}(x') = :\hat{\phi}(x)\hat{\phi}(x'): + \overline{\hat{\phi}(x)\hat{\phi}(x')} \ ,$$

where $:(\ldots):$ denotes the normal ordered product, and the contraction of two fields is defined as

$$\overline{\hat{\phi}(x)\hat{\phi}(x')} = \begin{cases} [\hat{\phi}^+(x), \hat{\phi}^-(x')], & \text{for } t > t' \\ [\hat{\phi}^+(x'), \hat{\phi}^-(x)], & \text{for } t' > t \end{cases} \equiv \mathcal{D}_F(x,x') \ ,$$

with $\mathcal{D}_F(x,x')$ the Feynman propagator.

2. Show that the four-point function can be written as

$$\begin{aligned} \left\langle 0\right| T\hat{\phi}(x_1)\hat{\phi}(x_2)\hat{\phi}(x_3)\hat{\phi}(x_4) \left|0\right\rangle = {} & \mathcal{D}_F(x_1,x_2)\mathcal{D}_F(x_3,x_4) \\ & + \mathcal{D}_F(x_1,x_3)\mathcal{D}_F(x_2,x_4) \\ & + \mathcal{D}_F(x_1,x_4)\mathcal{D}_F(x_2,x_3) \ . \end{aligned}$$

3. Wick's theorem for fermions

In the previous exercise you were asked to prove Wick's theorem for two and four scalar fields. Here you will prove Wick's theorem for an arbitrary number of fermions. Note that the time-ordered and normal-ordered products are modified to reflect the fermionic nature of the fields. For example,

$$T\psi(x_1)\psi(x_2)\psi(x_3)\psi(x_4) = (-1)^3 \psi(x_3)\psi(x_1)\psi(x_4)\psi(x_2) \qquad \text{if } x_3^0 > x_1^0 > x_4^0 > x_2^0 \ ,$$

and

$$:a_{\vec{p}} a_{\vec{q}} a_{\vec{r}}^+: = (-1)^2 a_{\vec{r}}^+ a_{\vec{p}} a_{\vec{q}} = (-1)^3 a_{\vec{r}}^+ a_{\vec{q}} a_{\vec{p}} \ .$$

The contraction of two fermionic fields is defined as

$$\contraction{}{\psi}{(x)}{\overline{\psi}}\psi(x)\overline{\psi}(y) = \begin{cases} \{\psi^+(x), \overline{\psi}^-(y)\}, & \text{for } x^0 > y^0 \\ -\{\overline{\psi}^+(y), \psi^-(x)\}, & \text{for } x^0 < y^0 \end{cases} \equiv \mathcal{S}_F(x,y) \ ,$$

and

$$\contraction{}{\psi}{(x)}{\psi}\psi(x)\psi(y) = \contraction{}{\overline{\psi}}{(x)}{\overline{\psi}}\overline{\psi}(x)\overline{\psi}(y) = 0 \ ,$$

with $\mathcal{S}_F(x,y)$ the Feynman propagator for the Dirac field. We also define contractions inside normal-ordered products to include minus signs for operator interchanges, such as

$$:\contraction{}{\psi}{(x_1)\psi(x_2)}{\overline{\psi}}\psi(x_1)\psi(x_2)\overline{\psi}(x_3)\overline{\psi}(x_4): \, = -\contraction{}{\psi}{(x_1)}{\overline{\psi}}\psi(x_1)\overline{\psi}(x_3) :\psi(x_2)\overline{\psi}(x_4): \, = -\mathcal{S}_F(x_1,x_3) :\psi(x_2)\overline{\psi}(x_4): \ .$$

Using the above, prove that

$$T\psi(x_1)\overline{\psi}(x_2)\psi(x_3)\cdots = \, :\psi(x_1)\overline{\psi}(x_2)\psi(x_3)\cdots + \text{all possible contractions}: \ .$$