(1) $\psi \to \psi' = e^{i\alpha\gamma_5}\psi$

$\rightsquigarrow \psi^+ \to \psi^{+\prime} = (e^{i\alpha\gamma_5}\psi)^+ = \psi^+ e^{-i\alpha\gamma_5^+} = \psi^+ e^{-i\alpha\gamma_5}$,

since $\gamma_5^+ = \gamma_5$

$\to \bar{\psi}' = (\psi^+\gamma^0)' = \psi^+ e^{-i\alpha\gamma_5}\gamma^0 = \psi^+ \sum_{n=0}^{+\infty} \frac{(-i\alpha\gamma_5)^n}{n!}\gamma^0 = \dots$

$= \psi^+(\cos\alpha - i\gamma_5 \sin\alpha)\gamma^0 = \bar{\psi}(\cos\alpha + i\gamma_5 \sin\alpha)$

$= \dots = \bar{\psi} \sum_{n=0}^{+\infty} \frac{(i\alpha\gamma_5)^n}{n!}$

$\to \bar{\psi}' = \bar{\psi} e^{i\alpha\gamma_5}$.

Then $S' = \int d^4x \, [i\bar{\psi}'\gamma^\mu \partial_\mu \psi' - m\bar{\psi}'\psi']$

$= \int d^4x \, [i\bar{\psi} e^{i\alpha\gamma_5}\gamma^\mu e^{i\alpha\gamma_5} \partial_\mu \psi - m\bar{\psi} e^{2i\alpha\gamma_5}\psi]$

Since $e^{i\alpha\gamma_5}\gamma^\mu e^{i\alpha\gamma_5} = \gamma^\mu$, we find

$S' = \int d^4x \, [i\bar{\psi}\gamma^\mu \partial_\mu \psi - m\bar{\psi} e^{2i\alpha\gamma_5}\psi]$

$\to S' = S$ only if $m = 0$.

To find the Noether current, we consider the infinitesimal transformation

$$\delta_5 \psi = i\alpha\gamma_5\psi, \quad \delta_5\bar{\psi} = i\alpha\bar{\psi}\gamma_5.$$

Then $$j_5^\mu = \frac{\delta \mathcal{L}}{\delta \partial_\mu \psi}\delta_5\psi + \delta_5\bar{\psi}\cancel{\frac{\delta \mathcal{L}}{\delta \partial_\mu \bar{\psi}}}^{\,0}$$

$$\rightarrow \boxed{j_5^\mu = -\bar{\psi}\gamma^\mu\gamma_5\psi}$$

$\partial_\mu j_5^\mu = -\partial_\mu(\bar{\psi}\gamma^\mu\gamma_5\psi) = 0$ on the equations of motion.

(2) We have

$$P_{L,R} = \frac{1}{2}(1 \mp \gamma_5)$$

Therefore: $P_L^2 = \frac{1}{4}(1-\gamma_5)^2 = \frac{1}{2}(1-\gamma_5) = P_L$

$$P_R^2 = \frac{1}{4}(1+\gamma_5)^2 = \frac{1}{2}(1+\gamma_5) = P_R$$

$$P_L P_R = P_R P_L = \frac{1}{4}(1-\gamma_5^2) = 0$$

$$P_L + P_R = P_R + P_L = 1.$$

(3) Since $P_L + P_R = 1$

$\rightarrow \psi = (P_L + P_R)\psi = \psi_L + \psi_R$

Therefore:

$$S' = \int d^4x \left[ i \bar{\psi} \gamma^\mu \partial_\mu \psi_L + i \bar{\psi} \gamma^\mu \partial_\mu \psi_R - m \bar{\psi} \psi_L - m \bar{\psi} \psi_R \right]$$

Using the fact that $P_{L,R}^2 = P_{L,R}$

$$\rightarrow S' = \int d^4x \left[ i \left( \bar{\psi} \gamma^\mu P_L \partial_\mu \psi_L + \bar{\psi} \gamma^\mu P_R \partial_\mu \psi_R \right) - m \left( \bar{\psi} P_L \psi_L + \bar{\psi} P_R \psi_R \right) \right]$$

From the anticommutation relations between $\gamma_5$ and $\gamma^\mu$ together with the explicit definitions of $P_L, P_R$, $\bar{\psi}$, one can show that

$$\gamma^\mu P_{L,R} = P_{R,L} \gamma^\mu \ , \quad \bar{\psi} P_{L,R} = \bar{\psi}_{R,L}$$

$$\rightarrow S' = \int d^4x \left[ i \bar{\psi}_L \gamma^\mu \partial_\mu \psi_L + i \bar{\psi}_R \gamma^\mu \partial_\mu \psi_R - m \left( \bar{\psi}_R \psi_L + \bar{\psi}_L \psi_R \right) \right]$$

Notice that for $m = 0$, there is no mixing between $\psi_L$ & $\psi_R$.

4 We know $\psi' = e^{i\alpha\gamma_5} \psi$. Then,

$\psi'_{L,R} = P_{L,R} e^{i\alpha\gamma_5} \psi$. Using $P_{L,R} \gamma_5 = \mp P_{L,R}$

$$\rightarrow \psi'_{L,R} = e^{\mp i\alpha} \psi_{L,R}$$

1. Plane-wave solutions of the Dirac equation

① ⓐ Starting from $(i\partial\!\!\!/ - m)\psi = 0$ & plugging

$\psi = e^{-ipx} u(\vec{p})$, we obtain

$$(p\!\!\!/ - m)u(\vec{p}) = \begin{pmatrix} p^0 - m & -\vec{p}\cdot\vec{\sigma} \\ \vec{p}\cdot\vec{\sigma} & -p^0 - m \end{pmatrix} u(\vec{p}) = 0 . \quad \circledast$$

Notice that

$$\det\begin{pmatrix} p^0 - m & -\vec{p}\cdot\vec{\sigma} \\ \vec{p}\cdot\vec{\sigma} & -p^0 - m \end{pmatrix} = 0, \text{ implying that}$$

$$-(p^0)^2 + m^2 + (\vec{p}\cdot\vec{\sigma})^2 = -(p_\mu p^\mu)^2 + m^2 = 0$$

$$\rightarrow p_\mu p^\mu = m^2$$

---

ⓑ Writing $u(\vec{p}) = \begin{pmatrix} \xi(\vec{p}) \\ \eta(\vec{p}) \end{pmatrix}$, we find that $\circledast$ gives the following set of equations:

$$(p^0 - m)\,\xi(\vec{p}) = \vec{p}\cdot\vec{\sigma}\,\eta(\vec{p}), \qquad (1)$$

$$\eta(\vec{p}) = \frac{\vec{p}\cdot\vec{\sigma}}{p^0 + m}\,\xi(\vec{p}) \qquad (2)$$

---

Plugging (2) into (1), we find that it is identically satisfied, meaning that the 2-component spinor $\xi(\vec{p})$ is arbitrary. Introducing $\chi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\chi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, we see that the Dirac equation has two solutions

$$u_s(\vec{p}) = \sqrt{p^0+m} \begin{pmatrix} \chi_s \\ \frac{\vec{\sigma}\cdot\vec{p}}{p^0+m}\chi_s \end{pmatrix}, \quad s = 1, 2.$$

② for $\psi = e^{ipx} v(\vec{p})$, we find

$$v_s(\vec{p}) = -\sqrt{p^0+m} \begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{p^0+m}\varepsilon\chi_s \\ \varepsilon\chi_s \end{pmatrix}, \quad s = 1, 2, \quad \&$$

$$\varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

③ We have from the previous point that

$$u_1(\vec{p}) = \sqrt{p^0+m} \begin{pmatrix} \chi_1 \\ \frac{\vec{\sigma}\cdot\vec{p}}{p^0+m}\chi_1 \end{pmatrix} = \sqrt{p^0+m} \begin{pmatrix} 1 \\ 0 \\ \frac{p_3}{p^0+m} \\ \frac{p_1+ip_2}{p^0+m} \end{pmatrix},$$

meaning that

$$\bar{u}_1(\vec{p}) \equiv u_1^\dagger(\vec{p})\gamma^0 = \sqrt{p^0+m}\left(1, 0, -\frac{p_3}{p^0+m}, \frac{-p_1+ip_2}{p^0+m}\right)$$

Therefore

$$u_1(\vec{p})\,\bar{u}_1(\vec{p}) = \begin{pmatrix} p^0+m & 0 & -p_3 & -(p_1-ip_2) \\ 0 & 0 & 0 & 0 \\ p_3 & 0 & -\frac{p_3^2}{p^0+m} & -\frac{p_3(p_1-ip_2)}{p^0+m} \\ p_1+ip_2 & 0 & -\frac{p_3(p_1+ip_2)}{p^0+m} & -\frac{p_1^2+p_2^2}{p^0+m} \end{pmatrix} \quad (*)$$

Similarly,

$$u_2(\vec{p})\bar{u}_2(\vec{p}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & p^0+m & -(p_1+ip_2) & p_3 \\ 0 & p_1-ip_2 & -\frac{p_1^2+p_2^2}{p^0+m} & \frac{p_3(p_1-ip_2)}{p^0+m} \\ 0 & p_3 & \frac{p_3(p_1+ip_2)}{p^0+m} & -\frac{p_3^2}{p^0+m} \end{pmatrix} \quad (**)$$

From $(*)+(**)$, we easily find

$$\sum_s u_s(\vec{p})\bar{u}_s(\vec{p}) = \begin{pmatrix} p^0+m & 0 & -p_3 & -(p_1-ip_2) \\ 0 & p_0+m & -(p_1+ip_2) & p_3 \\ p_3 & p_1-ip_2 & -p^0+m & 0 \\ p_1+ip_2 & -p_3 & 0 & -p^0+m \end{pmatrix}$$

$$\sum_s u_s(\vec{p})\bar{u}_s(\vec{p}) = \not{p}+m$$

Using exactly the same logic, one can show that the completeness relation for $v_s(\vec{p})$ reads

$$\sum_s v_s(\vec{p})\bar{v}_s(\vec{p}) = \not{p}-m.$$

(4) $(*)$ let's consider first $s=r=1$

$$\bar{u}_1(\vec{p})u_1(\vec{p}) = (p^0+m)\left(1, 0, -\frac{p_3}{p^0+m}, \frac{-p_1+ip_2}{p^0+m}\right)\begin{pmatrix} 1 \\ 0 \\ \frac{p_3}{p^0+m} \\ \frac{p_1+ip_2}{p^0+m} \end{pmatrix}$$

$$= (p^0+m)\left(1 - \frac{\vec{p}^2}{(p^0+m)^2}\right) = 2m$$

Similarly for $s=r=2$, we obtain

$$\rightarrow \bar{u}_2(\vec{p})u_2(\vec{p}) = 2m,$$

while for $s \neq r$, we see that the expression vanishes. Collecting everything together,

$$\bar{u}_s(\vec{p})\, u_r(\vec{p}) = 2m\,\delta_{sr}\,.$$

* for $v_s(\vec{p})$, the computation goes exactly as before. The result reads

$$\bar{v}_s(\vec{p})\, v_r(\vec{p}) = -2m\,\delta_{sr}\,.$$

# Quantization of Dirac field.

We are given that the mode expansion of the Dirac field is given by

$$\psi(x) = \int \frac{d^3\vec{p}}{(2\pi)^3 2\omega_{\vec{p}}} \sum_s \left( u_s(\vec{p}) a_s(\vec{p}) e^{-ip\cdot x} + v_s(\vec{p}) b_s^\dagger(\vec{p}) e^{ip\cdot x} \right) \quad (1)$$

meaning that

$$\bar{\psi}(x) = \int \frac{d^3\vec{p}}{(2\pi)^3 2\omega_{\vec{p}}} \sum_s \left( \bar{v}_s(\vec{p}) b_s(\vec{p}) e^{-ip\cdot x} + \bar{u}_s(\vec{p}) a_s^\dagger(\vec{p}) e^{ipx} \right. \quad (2)$$

The Hamiltonian you have already found in a previous PS; it reads

$$H = \int d^3\vec{x}\, \bar{\psi}(-i\gamma^i \partial_i \psi + m)\psi \quad . \quad (3)$$

First we'll compute

$\circledast \cdot (-i\gamma^i\partial_i + m)\psi =$

$$(-i\gamma^i\partial_i + m) \int \frac{d^3\vec{p}}{(2\pi)^3 2\omega_{\vec{p}}} \sum_s \left( u_s(\vec{p}) a_s(\vec{p}) e^{-ipx} \right.$$

$$+ v_s(\vec{p}) b_s^\dagger(\vec{p}) e^{ipx} ] \quad (4)$$

Since

$$p \cdot x = p_\mu x^\mu = \omega_{\vec{p}} t + p_j x^j \left(= \omega_{\vec{p}} t - \vec{p}\vec{x}\right), \quad (5)$$

we obtain

$$\partial_i e^{-ipx} = \frac{\partial}{\partial x^i} e^{-i\omega_{\vec{p}} t - i p_j x^j} = -i p_i e^{-ipx}, \quad (6)$$

&

$$\partial_i e^{ipx} = \frac{\partial}{\partial x^i} e^{i\omega_{\vec{p}} t + i p_j x^j} = i p_i e^{ipx} \quad (7)$$

meaning that

$$\otimes = \int \frac{d^3p}{(2\pi)^3 2\omega_{\vec{p}}} \sum_s \left[ (-\gamma^i p_i + m) u_s(\vec{p}) a_s(\vec{p}) e^{-ipx} \right. \quad (8)$$

$$\left. + (\gamma^i p_i + m) v_s(\vec{p}) b_s^\dagger(\vec{p}) e^{ipx} \right]$$

The Dirac eq. in momentum space translates into

$$(-\gamma^i p_i + m) u_s(\vec{p}) = \gamma^0 \omega_{\vec{p}} u_s(\vec{p}), \quad (9)$$

$$(\gamma^i p_i + m) v_s(\vec{p}) = -\gamma^0 \omega_{\vec{p}} v_s(\vec{p}), \quad (10)$$

Plugging these into (8), we obtain for $\oplus$

$$\oplus = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{\gamma_0}{2} \sum_s \left[ u_s(\vec{p}) a_s(\vec{p}) e^{-ipx} - v_s(\vec{p}) b_s^+(\vec{p}) e^{ipx} \right] \quad (11)$$

With the above results, we find that the Hamiltonian (3) becomes

$$H = \int d^3\vec{x} \int \frac{d^3\vec{p}}{(2\pi)^3} \left[ \bar{\psi} \frac{\gamma_0}{2} \sum_s \left[ u_s(\vec{p}) a_s(\vec{p}) e^{-ipx} - v_s(\vec{p}) b_s^+(\vec{p}) e^{ipx} \right] \right.$$

$$= \frac{1}{2} \int d^3\vec{x} \int \frac{d^3\vec{p}}{(2\pi)^3} \int \frac{d^3\vec{p}'}{(2\pi)^3 2\omega_{p'}} \sum_{s,s'} \left[ \underline{\bar{v}_{s'}}(\vec{p}') b_{s'}(\vec{p}') e^{-ip'x} \right.$$

$$\left. + \underline{\bar{u}_{s'}}(\vec{p}') a_{s'}^+(\vec{p}') e^{ip'x} \right] \underline{\gamma_0} \left[ u_s(\vec{p}) a_s(\vec{p}) e^{-ipx} - v_s(\vec{p}) b_s^+(\vec{p}) e^{ipx} \right]$$

$$= \frac{1}{2} \int d^3\vec{x} \int \frac{d^3\vec{p}}{(2\pi)^3} \int \frac{d^3\vec{p}'}{(2\pi)^3 2\omega_{p'}} \sum_{s,s'} \left[ v_{s'}^+(\vec{p}') b_{s'}(\vec{p}') e^{-ip'x} \right.$$

$$\left. + u_{s'}^+(\vec{p}') a_{s'}^+(\vec{p}') e^{ip'x} \right] \left[ u_s(\vec{p}) a_s(\vec{p}) e^{-ipx} - v_s(\vec{p}) b_s^+(\vec{p}) e^{ipx} \right]$$

$$= \frac{1}{2}\int\frac{d^3\vec{p}}{(2\pi)^3}\int\frac{d^3\vec{p}\,'}{2\omega_{\vec{p}\,'}}\times$$

$$\times\sum_{s,s'}\Big[v^\dagger_{s'}(\vec{p}\,')u_s(\vec{p})b_{s'}(\vec{p}\,')a_s(\vec{p})\,e^{-i(\omega_{\vec{p}}+\omega_{\vec{p}\,'})t}\,\delta^{(3)}(\vec{p}+\vec{p}\,')$$

$$-v^\dagger_{s'}(\vec{p}\,')v_s(\vec{p})b_{s'}(\vec{p}\,')b^\dagger_s(\vec{p})\,e^{-i(\omega_{\vec{p}}-\omega_{\vec{p}\,'})t}\,\delta^{(3)}(\vec{p}-\vec{p}\,')$$

$$+u^\dagger_{s'}(\vec{p}\,')u_s(\vec{p})a^\dagger_{s'}(\vec{p}\,')a_s(\vec{p})\,e^{i(\omega_{\vec{p}}-\omega_{\vec{p}\,'})t}\,\delta^{(3)}(\vec{p}-\vec{p}\,')$$

$$-u^\dagger_{s'}(\vec{p}\,')v_s(\vec{p})a^\dagger_{s'}(\vec{p}\,')b^\dagger_s(\vec{p})\,e^{i(\omega_{\vec{p}}+\omega_{\vec{p}\,'})t}\,\delta^{(3)}(\vec{p}+\vec{p}\,')\Big]$$

$$=\frac{1}{2}\int\frac{d^3\vec{p}}{(2\pi)^3 2\omega_{\vec{p}}}\sum_{s,s'}\Big[v^\dagger_{s'}(-\vec{p})u_s(\vec{p})b_{s'}(-\vec{p})a_s(\vec{p})\,e^{-2i\omega_{\vec{p}}t}$$

$$-v^\dagger_{s'}(\vec{p})v_s(\vec{p})b_{s'}(\vec{p})b^\dagger_s(\vec{p})+u^\dagger_{s'}(\vec{p})u_s(\vec{p})a^\dagger_{s'}(\vec{p})a_s(\vec{p})$$

$$-u^\dagger_{s'}(-\vec{p})v_s(\vec{p})a^\dagger_{s'}(-\vec{p})b^\dagger_s(\vec{p})\,e^{2i\omega_{\vec{p}}t}\Big] \quad (12)$$

One can show explicitely that

$$v^\dagger_{s'}(-\vec{p})u_s(\vec{p}) = u^\dagger_{s'}(-\vec{p})v_s(\vec{p}) = 0,$$

$$v^\dagger_{s'}(-\vec{p})v_s(\vec{p}) = u^\dagger_{s'}(-\vec{p})u_s(\vec{p}) = 2\omega_{\vec{p}}\,\delta_{ss'} \quad (13)$$

Using the above, we get

$$H = \int \frac{d^3\vec{p}}{(2\pi)^3 2\omega_{\vec{p}}} \sum_s \omega_{\vec{p}} \left[ a_s^\dagger(\vec{p}) a_s(\vec{p}) - b_s(\vec{p}) b_s^\dagger(\vec{p}) \right] \quad (14)$$

① $$H = \int \frac{d^3\vec{p}}{(2\pi)^3 2\omega_{\vec{p}}} \sum_s \omega_{\vec{p}} \left[ a_s^\dagger(\vec{p}) a_s(\vec{p}) - b_s^\dagger(\vec{p}) b_s(\vec{p}) \right] + \infty \quad (15)$$

problem!!!

In going from (14) to (15), we used commutation relations.

② Assume anticommutation relations,

$$\{a_i(\vec{p}), a_j^\dagger(\vec{p}\,')\} = \{b_i(\vec{p}), b_j^\dagger(\vec{p}\,')\} \quad (16)$$
$$= (2\pi)^3 2\omega_{\vec{p}}\, \delta_{ij}\, \delta^{(3)}(\vec{p}-\vec{p}\,')$$

② then from (14) we get

$$H = \int \frac{d^3\vec{p}}{(2\pi)^3 2\omega_{\vec{p}}} \sum_s \omega_{\vec{p}} \left[ a_s^\dagger(\vec{p}) a_s(\vec{p}) + b_s^\dagger(\vec{p}) b_s(\vec{p}) \right] \quad (17)$$

(b) $Q = \int d^3\vec{x}\, j^0 = -\int d^3\vec{x}\, \bar{\psi}\gamma^0\psi$

(18)

$\rightsquigarrow Q = -\int d^3\vec{x}\, \psi^\dagger\psi$

Then from (1) & (2) it's easy to show that

$$Q = -\int \frac{d^3\vec{p}}{(2\pi)^3 2\omega_{\vec{p}}} \sum_s [a_s^\dagger(\vec{p})a_s(\vec{p}) - b_s^\dagger(\vec{p})b_s(\vec{p})]$$

(c) $a_i^\dagger(\vec{p})a_j^\dagger(\vec{p}\,')|0\rangle \equiv |\vec{p}, i\,; \vec{p}\,', j\rangle$

$= -a_j^\dagger(\vec{p}\,')a_i^\dagger(\vec{p})|0\rangle = |\vec{p}\,'; j\,; \vec{p}, i\rangle$

antisymmetric under the exchange of the particles $\longleftrightarrow$ Fermi-Dirac statistics

for $\vec{p} = \vec{p}\,'$ & $i = j$ $\to$

$|\vec{p}, i\,; \vec{p}, i\rangle = a_i^\dagger(\vec{p})a_i^\dagger(\vec{p}\,')|0\rangle = (a_i^\dagger(\vec{p}))^2|0\rangle$

$= 0$

$\to$ Pauli exclusion principle.

$$\otimes \quad H|\vec{p},i;\vec{p}\,',j\rangle = \int \frac{d^3\vec{k}}{(2\pi)^3 2\omega_{\vec{k}}} \sum_s \omega_{\vec{k}} \Big[ a_s^{\dagger}(\vec{k}) a_s(\vec{k})$$

$$+ b_s^{\dagger}(\vec{k}) b_s(\vec{k}) \Big] a_i^{\dagger}(\vec{p}) a_j^{\dagger}(\vec{p}\,')|0\rangle$$

$$= \int \frac{d^3\vec{k}}{(2\pi)^3 2\omega_{\vec{k}}} \sum_s \omega_{\vec{k}}\, a_s^{\dagger}(\vec{k}) a_s(\vec{k})\, a_i^{\dagger}(\vec{p}) a_j^{\dagger}(\vec{p}\,')|0\rangle$$

Use $\quad \{a_s(\vec{k}), a_i^{\dagger}(\vec{p})\} = (2\pi)^3 2\omega_{\vec{k}}\, \delta_{si}\, \delta^{(3)}(\vec{k}-\vec{p})$

$$\rightarrow H|\vec{p},i;\vec{p}\,',j\rangle = \int \frac{d^3\vec{k}}{(2\pi)^3 2\omega_{\vec{k}}} \Big[ -\sum_s \omega_{\vec{k}}\, a_s^{\dagger}(\vec{k})\, a_i^{\dagger}(\vec{p})\, a_s(\vec{k})\, a_j^{\dagger}(\vec{p}\,')|0\rangle$$

$$+ \omega_{\vec{k}} (2\pi)^3 2\omega_{\vec{k}}\, \delta^{(3)}(\vec{k}-\vec{p})\, a_i^{\dagger}(\vec{k}) a_j^{\dagger}(\vec{p}\,')|0\rangle$$

$$= \int \frac{d^3\vec{k}}{\cancel{(2\pi)^3 2\omega_{\vec{k}}}} \Big[ -\omega_{\vec{k}} \cancel{(2\pi)^3 2\omega_{\vec{k}}}\, a_j^{\dagger}(\vec{k})\, a_i^{\dagger}(\vec{p})|0\rangle \delta^{(3)}(\vec{k}-\vec{p}\,')$$

$$+ \omega_{\vec{k}} \cancel{(2\pi)^3 2\omega_{\vec{k}}}\, a_i^{\dagger}(\vec{k})\, a_j^{\dagger}(\vec{p}\,')|0\rangle \delta^{(3)}(\vec{k}-\vec{p}) \Big]$$

$$= -\omega_{\vec{p}\,'}\, a_j^{\dagger}(\vec{p}\,') a_i^{\dagger}(\vec{p})\, |0\rangle$$

$$+ \omega_{\vec{p}}\, a_i^{\dagger}(\vec{p}) a_j^{\dagger}(\vec{p}\,')\, |0\rangle$$

$$= (\omega_{\vec{p}} + \omega_{\vec{p}\,'})\, a_i^{\dagger}(\vec{p})\, a_j^{\dagger}(\vec{p}\,')|0\rangle .$$

$$Q|\vec{p}, i; \vec{p}', j\rangle = -\int \frac{d^3\vec{k}}{(2\pi)^3 2\omega_{\vec{k}}} \sum_s a_s^+(\vec{k})\, a_s(\vec{k})\, a_i^+(\vec{p})\, a_j^+(\vec{p}')|0\rangle$$

$$= -\int \frac{d^3\vec{k}}{(2\pi)^3 2\omega_{\vec{k}}} \Big[ -\sum_s a_s^+(\vec{k}) a_i^+(\vec{p})\, a_s(\vec{k})\, a_j^+(\vec{p}')|0\rangle$$

$$+ (2\pi)^3 2\omega_{\vec{k}}\; a_i^+(\vec{k})\, a_j^+(\vec{p}')|0\rangle \delta^{(3)}(\vec{k}-\vec{p}) \Big]$$

$$= -\int \frac{d^3\vec{k}}{(2\pi)^3 2\omega_{\vec{k}}} \Big[ -\cancel{(2\pi)^3 2\omega_{\vec{k}}}\; a_j^+(\vec{k}) a_i^+(\vec{p})|0\rangle \, \delta^{(3)}(\vec{k}-\vec{p}')$$

$$+ \cancel{(2\pi)^3 2\omega_{\vec{k}}}\; a_i^+(\vec{k})\, a_j^+(\vec{p}')|0\rangle \delta^{(3)}(\vec{k}-\vec{p})$$

$$= \left( a_j^+(\vec{p}')\, a_i^+(\vec{p}) \;-\; a_i^+(\vec{p})\, a_j^+(\vec{p}')\right)|0\rangle$$

$$= -2 a_i^+(\vec{p})\, a_j^+(\vec{p}')\,|0\rangle .$$

Similarly

$$Q\, b^+(\vec{p}) b_j^+(\vec{p}')|0\rangle = 2 b_i^+(\vec{p})\, b_j^+(\vec{p}')|0\rangle .$$