Quantum Field Theory (Quantum Electrodynamics)

Problem Set 7 4 & 6 December 2023

1. The Dirac field, continued

Consider

$$S = \int \mathrm{d}^4x \left(i\bar{\psi}\gamma^\mu \partial_\mu \psi - m\bar{\psi}\psi \right) \ , \tag{1}$$

where $\bar{\psi} = \psi^\dagger \gamma^0$, and m is the (real) mass of the field.

1. Consider the transformation

$$\psi \to \psi' = \mathrm{e}^{i\alpha\gamma^5}\psi \ , \tag{2}$$

   with α a nonzero real constant and $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$. What do we have to require in order for the action (1) to be invariant under the above transformation ? Construct the Noether current and verify that it is conserved on the equations of motion.
2. Check that the operators

$$P_{L,R} = \frac{1 \mp \gamma^5}{2},$$

   satisfy

$$P_L^2 = P_L \ , \quad P_R^2 = P_R \ , \quad P_L P_R = 0 \ , \quad P_R P_L = 0 \ , \quad P_L + P_R = 1 \ ,$$

   i.e. they are projection operators.
3. Introduce the left- and right- handed fields $\psi_{L,R} = P_{L,R}\psi$ and rewrite the action (1) in the terms of $\psi_{L,R}$.
4. How do $\psi_{L,R}$ transform under (2) ?

2. Plane-wave solutions of the Dirac equation

The Dirac equation reads

$$(i\partial\!\!\!/ - m)\psi = 0 \ , \tag{3}$$

where $\partial\!\!\!/ = \gamma^\mu \partial_\mu$. For concreteness, we will be working with the Dirac representation of the gamma matrices, see the previous Problem Set.

1. Let's assume that

$$\psi = e^{-ipx} u(\vec{p}) \ ,$$

   where $u(\vec{p})$ a four-component spinor that depends only on the three-momentum $\vec{p}$.

   (a) Plug the above into (3) and write the resulting equation in matrix form. Find the condition on the four-momentum which follows from this equation.

(b) Represent the four-component spinor in terms of two-component spinors, i.e. $u(\vec{p}) = (\xi(\vec{p}), \eta(\vec{p}))^T$ and show that the solutions to the Dirac equation read

$$u_s(\vec{p}) = \sqrt{p^0 + m} \begin{pmatrix} \chi_s \\ \frac{\vec{\sigma}\cdot\vec{p}}{p^0+m}\chi_s \end{pmatrix} \quad , \qquad s = 1, 2 \; ,$$

with $\chi_1 = (1,0)^T$ and $\chi_2 = (0,1)^T$.

2. Using the ansatz $\psi = e^{+ipx} v(\vec{p})$, repeat the above steps and show that now the solutions read

$$v_s(\vec{p}) = -\sqrt{p^0 + m} \begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{p^0+m}\epsilon\chi_s \\ \epsilon\chi_s \end{pmatrix} \quad , \qquad s = 1, 2 \; , \qquad \epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

with χ_1 and χ_2 given above.

3. Show that

$$\sum_s u_s(\vec{p})\bar{u}_s(\vec{p}) = \not{p} + m \; , \qquad \sum_s v_s(\vec{p})\bar{v}_s(\vec{p}) = \not{p} - m \; ,$$

where $\not{p} = \gamma^\mu p_\mu$.

4. Compute

$$\bar{u}_s(\vec{p}) u_r(\vec{p}) \; , \quad \bar{v}_s(\vec{p}) v_r(\vec{p}) \; .$$

3. Quantization of the Dirac Field

The mode expansion of the Dirac field can be written in terms of ladder operators as

$$\psi(x) = \int \frac{d^3\vec{p}}{(2\pi)^3 2\omega_{\vec{p}}} \sum_i \left(u_i(\vec{p}) a_i(\vec{p}) e^{-ipx} + v_i(\vec{p}) b_i^+(\vec{p}) e^{ipx} \right) \; , \qquad (4)$$

1. Analogously to what we did for the scalar field, let us postulate the following commutation relations

$$[a_i(\vec{p}), a_j^+(\vec{p'})] = [b_i(\vec{p}), b_j^+(\vec{p'})] = (2\pi)^3 2\omega_{\vec{p}} \delta_{ij} \, \delta^{(3)}(\vec{p} - \vec{p'}) \; , \qquad (5)$$

and zero otherwise. Compute the normal-ordered Hamiltonian. Show that it is not bounded from bellow, meaning that quantizing the Dirac field by requiring that the ladder operators obey the commutation relations (5) is inconsistent.

2. The way out is to require that the ladder operators satisfy anticommutation relations

$$\{a_i(\vec{p}), a_j^+(\vec{p'})\} = \{b_i(\vec{p}), b_j^+(\vec{p'})\} = (2\pi)^3 2\omega_{\vec{p}} \delta_{ij} \, \delta^{(3)}(\vec{p} - \vec{p'}) \; , \qquad (6)$$

and zero otherwise.

(a) Compute the Hamiltonian of the Dirac field in terms of the ladder operators subject to (6). Is it bounded from bellow in this case?

(b) In the Problem Set 6 we saw that the Dirac theory is invariant under $\psi \to \psi' = e^{i\alpha}\psi$, and found that the corresponding Noether current reads $j^\mu = -\bar{\psi}\gamma^\mu\psi$. Write the charge $Q = \int d^3x j^0$ in terms of creation and annihilation operators.

(c) Consider the state $a_i^+(\vec{p}) a_j^+(\vec{p'}) |0\rangle$. What statistics does it obey? What is its energy and charge? What happens if both $\vec{p} = \vec{p'}$ and $i = j$?

(d) Find the charge of the state $b_i^+(\vec{p}) b_j^+(\vec{p'}) |0\rangle$.