Quantum Field Theory (Quantum Electrodynamics)

Problem Set 6 27 & 29 November 2023

1. The Poincaré group, continued

The commutation relations of the Poincaré group generators in a manifestly covariant form read as (see also Problem Set 5)

$$\begin{aligned} &[M_{\mu\nu}, M_{\rho\sigma}] = i\eta_{\mu\rho}M_{\nu\sigma} + i\eta_{\nu\sigma}M_{\mu\rho} - i\eta_{\mu\sigma}M_{\nu\rho} - i\eta_{\nu\rho}M_{\mu\sigma}, \\ &[M_{\mu\nu}, P_\rho] = i\eta_{\mu\rho}P_\nu - i\eta_{\nu\rho}P_\mu, \\ &[P_\mu, P_\nu] = 0. \end{aligned} \tag{1}$$

Define

$$J_i = \frac{1}{2}\epsilon_{ijk}M_{jk} \ , \quad K_i = M_{0i} \ ,$$

which correspond to the generators of spatial rotations and Lorentz boosts, respectively. Rewrite (1) in terms of J_i and K_i.

2. Gamma matrices in 4 spacetime dimensions

The γ-matrices satisfy

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathcal{I}_4, \tag{2}$$

where $\{a, b\} = [a, b]_+ = ab + ba$ is the anticommutator, and $\mathcal{I}_4 = \text{diag}(1, 1, 1, 1)$ is the 4×4 identity matrix. In what follows we will not be writing $\mathcal{I}_4$ explicitly, although its presence will be tacitly assumed.

1. In the Dirac representation, the form of the gamma matrices is

$$\gamma^0_D = \begin{pmatrix} \mathcal{I}_2 & 0 \\ 0 & -\mathcal{I}_2 \end{pmatrix} \ , \quad \gamma^i_D = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \ ,$$

with σ^i, $i = 1, 2, 3$ the Pauli matrices, and $\mathcal{I}_2 = \text{diag}(1, 1)$ the 2×2 identity matrix. Show that γ^μ_D satisfy (2).

In the Weyl or chiral representation, the γ-matrices read

$$\gamma^0_W = \begin{pmatrix} 0 & \mathcal{I}_2 \\ \mathcal{I}_2 & 0 \end{pmatrix} \ , \quad \gamma^i_W = \gamma^i_D \ .$$

Show that $\gamma^\mu_W = U\gamma^\mu_D U^+$, with

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathcal{I}_2 & -\mathcal{I}_2 \\ \mathcal{I}_2 & \mathcal{I}_2 \end{pmatrix} \ .$$

Compute $\{\gamma^\mu_W, \gamma^\nu_W\}$.

Hint : The matrix U is unitary, i.e. $U^+ = U^{-1}$.

2. Prove the following identities without using any representation of the γ matrices

$$\gamma^\mu\gamma_\mu = 4, \quad \gamma^\mu\gamma^\alpha\gamma_\mu = -2\gamma^\alpha, \quad \gamma^\mu\gamma^\alpha\gamma^\beta\gamma_\mu = 4\eta^{\alpha\beta},$$

$$\gamma^\mu\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma_\mu = -2\gamma^\gamma\gamma^\beta\gamma^\alpha,$$

$$\{\gamma^\mu, \gamma_5\} = 0, \quad (\gamma_5)^2 = 1, \quad \gamma_5^+ = \gamma_5, \quad \text{with} \quad \gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3 \ ,$$

$$\gamma^\mu\gamma^\nu\gamma^\rho = \eta^{\mu\nu}\gamma^\rho + \eta^{\nu\rho}\gamma^\mu - \eta^{\mu\rho}\gamma^\nu + i\epsilon^{\mu\nu\rho\sigma}\gamma^5\gamma_\sigma \ , \quad \text{with} \quad \epsilon_{0123} = -\epsilon^{0123} = 1.$$

3. Prove the following trace identities without using any representation of the γ matrices

$$\mathrm{tr}(\gamma^\mu\gamma^\nu) = 4\eta^{\mu\nu}, \quad \mathrm{tr}(\gamma_5) = \mathrm{tr}(\gamma_5\gamma^\mu\gamma^\nu) = 0,$$

$$\mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta) = 4(\eta^{\mu\nu}\eta^{\alpha\beta} + \eta^{\mu\beta}\eta^{\nu\alpha} - \eta^{\mu\alpha}\eta^{\nu\beta}),$$

$$\mathrm{tr}(\gamma_5\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta) = 4i\epsilon^{\mu\nu\alpha\beta}.$$

$$\mathrm{tr}(\text{odd number of } \gamma\text{'s}) = 0.$$

2. The Dirac field

Consider

$$S = \int \mathrm{d}^4x \left(i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi \right) \ , \tag{3}$$

where $\bar{\psi} = \psi^\dagger\gamma^0$, and m is the (real) mass of the field.

1. What is the dimension of ψ ?
2. Check that the action is Hermitian up to a total derivative.
   Useful formulas : $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0, \ (\gamma^0)^\dagger = \gamma^0, \ (\gamma^0)^2 = 1.$
3. Find the equations of motion for ψ and $\bar{\psi}$.
4. Check that the action is invariant under

$$\psi \to \psi' = e^{i\alpha}\psi \ ,$$

   with α a constant. Derive the corresponding Noether current. Is it conserved on the equations of motion ?
5. Find the energy-momentum tensor and verify its conservation.
6. Compute the canonical momenta π and $\bar{\pi}$.
7. Find the Hamiltonian and the three-momentum $\vec{P}$.