1. The Poincaré group, continued

Using

$$J_i = \frac{1}{2}\epsilon_{ijk} M_{jk} \quad , \quad (1)$$

we find

$$M_{ij} = \epsilon_{ijk} J_k \quad . \quad (2)$$

We also define

$$M_{0i} = K_i \quad . \quad (3)$$

The commutation relations become in terms of P_j

$$[M_{\mu\nu}, P_\rho] = i(\eta_{\mu\rho} P_\nu - \eta_{\nu\rho} P_\mu) \quad (4)$$

- $[M_{0i}, P_0] = [K_i, P_0] = i\,\eta_{00} P_i$

$$\Rightarrow \boxed{[K_i, P_0] = i P_i \; .} \quad (5)$$

- $[M_{0i}, P_j] = [K_i, P_j] = -i\eta_{ij} P_0$

$$\Rightarrow \boxed{[K_i, P_j] = i\delta_{ij} P_0} \; . \quad (6)$$

- $[M_{ij}, P_0] = 0$

$$\boxed{\Rightarrow [J_i, P_0] = 0 \; .} \quad (7)$$

- $[M_{ij}, P_k] = i(\eta_{ik} P_j - \eta_{jk} P_i) = -i(\delta_{ik} P_j - \delta_{jk} P_i)$

$$\Rightarrow \epsilon_{ijl}[J_l, P_k] = -i(\delta_{ik} P_j - \delta_{jk} P_i) \; . \quad (8)$$

multiply (8) by ϵ_{ijm} to get

$$\begin{aligned} 2[J_m, P_k] &= -i(\epsilon_{kjm} P_j - \epsilon_{ikm} P_i) \\ &= -i(\epsilon_{kim} P_i - \epsilon_{ikm} P_i) \\ &= 2i\epsilon_{ikm} P_i = -2i\epsilon_{imk} P_i = 2i\epsilon_{mik} P_i \\ &= -2i\epsilon_{mki} P_i \end{aligned}$$

Thus: $$\boxed{[J_i, P_j] = i\epsilon_{ijk} P_k} \quad (9)$$

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$\circledast\ [M_{\mu\nu}, M_{\rho\sigma}] = i(\eta_{\mu\rho} M_{\nu\sigma} + \eta_{\nu\sigma} M_{\mu\rho} - \eta_{\mu\sigma} M_{\nu\rho} - \eta_{\nu\rho} M_{\mu\sigma})$ (10)

- $[M_{0i}, M_{0j}] \equiv [K_i, K_j] = i\eta_{00} M_{ij} = i\epsilon_{ijk} J_k$

$\rightarrow$ $\boxed{[K_i, K_j] = i\epsilon_{ijk} J_k}$ (11)

- $[M_{0i}, M_{jk}] = \epsilon_{jkl}[K_i, J_l] = i(\eta_{ik} M_{0j} - \eta_{ij} M_{0k})$

$= -i(\delta_{ik} K_j - \delta_{ij} K_k)$

$\rightarrow \epsilon_{jkl}[K_i, J_l] = -i(\delta_{ik} K_j - \delta_{ij} K_k)$ (12)

multiply (12) by ϵ_{jkm}, to find

$$2\delta_{lm}[K_i, J_l] = -i(\epsilon_{jim} K_j - \epsilon_{ikm} K_k)$$

$$= -i(\epsilon_{jim} K_j - \epsilon_{ijm} K_j)$$

$$= -2i\epsilon_{imj} K_j$$

$\rightarrow [K_i, J_m] = -i\epsilon_{imj} K_j$

or in other words,

$[J_m, K_i] = \epsilon_{imj} K_j = -\epsilon_{mij} K_j$,

$\rightarrow$ $\boxed{[J_i, K_j] = -\epsilon_{ijk} K_k.}$ (13)

$\circledast\ [M_{ij}, M_{kl}] = \ldots = \ldots$

Explicitly:

$\epsilon_{ija}\epsilon_{klb}[J_a, J_b] = -i(\delta_{ik}\epsilon_{jla} + \delta_{jl}\epsilon_{ika} - \delta_{il}\epsilon_{jka} - \delta_{jk}\epsilon_{ila}) J_a$

multiply by $\epsilon_{ijc}\epsilon_{kld}$, to find

$\boxed{[J_a, J_b] = i\epsilon_{abc} J_c.}$ (14)

Gamma matrices in 4 spacetime dimensions

① In the Dirac representation,

$$\gamma_D^0 = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix}, \quad \gamma_D^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix},$$

with $1_2 = \mathrm{diag}(1,1)$ & σ^i the Pauli matrices.

⊗ $\{\gamma_D^0, \gamma_D^0\} = 2(\gamma_D^0)^2 = 2\,1_4 = 2\eta^{00}1_4$, since $\eta^{00} = 1$.

⊗ $\{\gamma_D^0, \gamma_D^i\} = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix}\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} + \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix} = 0$

⊗ $\{\gamma_D^i, \gamma_D^j\} = -\begin{pmatrix} \{\sigma^i, \sigma^j\} & 0 \\ 0 & \{\sigma^i, \sigma^j\} \end{pmatrix} = -2\delta^{ij}\begin{pmatrix} 1_2 & 0 \\ 0 & 1_2 \end{pmatrix}$

$$= 2\eta^{ij}1_4$$

Putting everything together, $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}1_4$.

In the Weyl representation,

$$\gamma_W^0 = \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix}, \quad \gamma_W^i = \gamma_D^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}.$$

We have $U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1_2 & -1_2 \\ 1_2 & 1_2 \end{pmatrix} \Rightarrow U^+ = \frac{1}{\sqrt{2}}\begin{pmatrix} 1_2 & 1_2 \\ -1_2 & 1_2 \end{pmatrix}$, so

$$\left.\begin{aligned} U\gamma_D^0 U^+ &= \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix} \equiv \gamma_W^0. \\ \text{Also,} \quad U\gamma_D^i U^+ &= \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \equiv \gamma_W^i \end{aligned}\right\} \Rightarrow \gamma_W^\mu = U\gamma_D^\mu U^+.$$

⊛ using $\gamma^\mu_W = U\gamma^\mu U^\dagger$, we find

$$\{\gamma^\mu_W, \gamma^\nu_W\} = \{U\gamma^\mu_D U^\dagger, U\gamma^\nu_D U^\dagger\}$$

$$= U\gamma^\mu_D U^\dagger U \gamma^\nu_D U^\dagger + U\gamma^\nu_D U^\dagger U \gamma^\mu_D U^\dagger$$

$$= U\{\gamma^\mu_D, \gamma^\nu_D\}U^\dagger = 2\eta^{\mu\nu}\mathbb{1}_4,$$

where we used the fact that U is a unitary matrix

② ⊛ $\gamma^\mu\gamma_\mu = \eta_{\mu\nu}\gamma^\mu\gamma^\nu = \frac{1}{2}(\eta_{\mu\nu} + \eta_{\nu\mu})\gamma^\mu\gamma^\nu$

$$= \frac{1}{2}(\eta_{\mu\nu}\gamma^\mu\gamma^\nu + \eta_{\mu\nu}\gamma^\nu\gamma^\mu) = \frac{1}{2}\eta_{\mu\nu}\{\gamma^\mu, \gamma^\nu\}$$

$$= \eta_{\mu\nu}\eta^{\mu\nu} = 4$$

⊛ $\gamma^\mu\gamma^\alpha\gamma_\mu = \gamma^\mu(\{\gamma^\alpha, \gamma_\mu\} - \gamma_\mu\gamma^\alpha) = \gamma^\mu(2\delta^\alpha_\mu - \gamma_\mu\gamma^\alpha)$

$$= -2\gamma^\alpha.$$

⊛ $\gamma^\mu\gamma^\alpha\gamma^\beta\gamma_\mu = \gamma^\mu\gamma^\alpha(\{\gamma^\beta, \gamma_\mu\} - \gamma_\mu\gamma^\beta) = \gamma^\mu\gamma^\alpha(2\delta^\beta_\mu - \gamma_\mu\gamma^\beta)$

$$= 2\gamma^\beta\gamma^\alpha - \gamma^\mu\gamma^\alpha\gamma_\mu\gamma^\beta = 2\gamma^\beta\gamma^\alpha - \gamma^\mu(\{\gamma^\alpha, \gamma_\mu\} - \gamma_\mu\gamma^\alpha)\gamma^\beta$$

$$= 2\gamma^\beta\gamma^\alpha - \gamma^\mu(2\delta^\alpha_\mu - \gamma_\mu\gamma^\alpha)\gamma^\beta = 2\{\gamma^\alpha, \gamma^\beta\}$$

$$= 4\eta^{\alpha\beta}$$

⊛ $\gamma^\mu\gamma^\alpha\gamma^\beta\gamma^\sigma\gamma_\mu = (\{\gamma^\mu, \gamma^\alpha\} - \gamma^\alpha\gamma^\mu)\gamma^\beta\gamma^\sigma\gamma_\mu = 2\gamma^\beta\gamma^\sigma\gamma^\alpha - \gamma^\alpha\underbrace{\gamma^\mu\gamma^\beta\gamma^\sigma\gamma_\mu}_{\text{previous identity!}}$

$$= 2\gamma^\beta\gamma^\sigma\gamma^\alpha - 4\gamma^\alpha\eta^{\beta\sigma}$$

$$= 2(\cancel{2\eta^{\beta\sigma}} - \gamma^\sigma\gamma^\beta)\gamma^\alpha - \cancel{4\gamma^\alpha\eta^{\beta\sigma}}$$

$$= -2\gamma^\sigma\gamma^\beta\gamma^\alpha.$$

⊛ Consider first $\underline{\mu = 0}$

- $\{\gamma^0, \gamma_5\} = i(\gamma_1\gamma_2\gamma_3 + \gamma_0\gamma_1\gamma_2\gamma_3\gamma^0) = i(\gamma_1\gamma_2\gamma_3 - \gamma_1\gamma_0\gamma_2\gamma_3\gamma^0)$

$= i(\gamma_1\gamma_2\gamma_3 + \gamma_1\gamma_2\gamma_0\gamma_3\gamma^0) = i(\gamma_1\gamma_2\gamma_3 - \gamma_1\gamma_2\gamma_3) = 0$

similarly, $\{\gamma^1, \gamma_5\} = \{\gamma^2, \gamma_5\} = \{\gamma^3, \gamma_5\} = 0$

$\rightarrow \underline{\{\gamma^\mu, \gamma_5\} = 0}$.

⊛ $(\gamma_5)^2 = -\gamma_0\gamma_1\gamma_2\gamma_3\gamma_0\gamma_1\gamma_2\gamma_3 = \gamma_1\gamma_0\gamma_2\gamma_3\gamma_0\gamma_1\gamma_2\gamma_3 = -\gamma_1\gamma_2\gamma_0\gamma_3\gamma_0\gamma_1\gamma_2\gamma_3$

$= \gamma_1\gamma_2\gamma_3\gamma_1\gamma_2\gamma_3 = -\gamma_2\gamma_1\gamma_3\gamma_1\gamma_2\gamma_3 = -\gamma_2\gamma_3\gamma_2\gamma_3 = 1$

⊛ $\gamma_5^\dagger = -i\gamma_3^\dagger\gamma_2^\dagger\gamma_1^\dagger\gamma_0^\dagger = i\gamma_3\gamma_2\gamma_1\gamma_0 = -i\gamma_3\gamma_2\gamma_0\gamma_1 = i\gamma_3\gamma_0\gamma_2\gamma_1$

$= -i\gamma_0\gamma_3\gamma_2\gamma_1 = i\gamma_0\gamma_3\gamma_1\gamma_2 = -i\gamma_0\gamma_1\gamma_3\gamma_2 = i\gamma_0\gamma_1\gamma_2\gamma_3$

$= \underline{\gamma_5}$.

3) ⊛ $tr(\gamma^\mu\gamma^\nu) = tr\left[\frac{1}{2}(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu)\right] = \frac{1}{2} tr\{\gamma^\mu, \gamma^\nu\}$

$= Tr(\mathbb{1}_4)\eta^{\mu\nu} = 4\eta^{\mu\nu}$

⊛ $tr(\gamma_5) = i\,tr(\gamma_0\gamma_1\gamma_2\gamma_3) = i\,tr(\gamma_1\gamma_2\gamma_3\gamma_0)$. At the same time,

$tr(\gamma_5) = -i\,tr(\gamma_1\gamma_0\gamma_2\gamma_3) = i\,tr(\gamma_1\gamma_2\gamma_0\gamma_3) = -i\,tr(\gamma_1\gamma_2\gamma_3\gamma_0) = -tr(\gamma_5)$

$\rightarrow tr(\gamma_5) = 0$.

⊛ To show $tr(\gamma_5\gamma^\mu\gamma^\nu) = 0$, it's simpler to consider different cases:

- $\underline{\mu=\nu=0}$ $\rightsquigarrow$ $tr(\gamma_5\gamma^0\gamma^0) = tr(\gamma_5) = 0$

- $\underline{\mu=\nu\neq 0}$ $\rightarrow$ $tr(\gamma_5\gamma^i\gamma^i) = -tr(\gamma_5) = 0$

- $\underline{\mu=0, \nu=1}$ $\rightarrow$ $tr(\gamma_5\gamma^0\gamma^1) = i\,tr(\gamma_0\gamma_1\gamma_2\gamma_3\gamma^0\gamma^1)$

$$= i\,tr(\gamma^0\gamma^1\gamma_0\gamma_1\gamma_2\gamma_3) = -i\,tr(\gamma_2\gamma_3) = 0 .$$

$\vdots$

$\rightarrow$ $\underline{tr(\gamma_5\gamma^\mu\gamma^\nu) = 0}$.

⊛ $tr(\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta) = tr[\gamma^\mu\{\gamma^\nu,\gamma^\alpha\}\gamma^\beta - \gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta]$

$$= 8\eta^{\mu\beta}\eta^{\nu\alpha} - tr(\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta)$$

$$= 8\eta^{\mu\beta}\eta^{\nu\alpha} - tr(\gamma^\mu\gamma^\alpha\{\gamma^\nu,\gamma^\beta\} - \gamma^\mu\gamma^\alpha\gamma^\beta\gamma^\nu)$$

$$= 8\eta^{\mu\beta}\eta^{\nu\alpha} - 8\eta^{\mu\alpha}\eta^{\nu\beta} + tr(\gamma^\mu\gamma^\alpha\gamma^\beta\gamma^\nu)$$

$$= 8\eta^{\mu\beta}\eta^{\nu\alpha} - 8\eta^{\mu\alpha}\eta^{\nu\beta} + tr(\gamma^\nu\gamma^\mu\gamma^\alpha\gamma^\beta)$$

$$= 8\eta^{\mu\beta}\eta^{\nu\alpha} - 8\eta^{\mu\alpha}\eta^{\nu\beta} + tr(\{\gamma^\nu,\gamma^\mu\}\gamma^\alpha\gamma^\beta - \gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta)$$

$$= 8\eta^{\mu\beta}\eta^{\nu\alpha} - 8\eta^{\mu\alpha}\eta^{\nu\beta} + 8\eta^{\mu\nu}\eta^{\alpha\beta} - tr(\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta)$$

$\rightarrow$ $tr(\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta) = 4(\eta^{\mu\nu}\eta^{\alpha\beta} + \eta^{\mu\beta}\eta^{\nu\alpha} - \eta^{\mu\alpha}\eta^{\nu\beta})$

⊛ We notice that $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$

Since the γ-matrices anticommute,

$\gamma_5 = i\gamma_{[0}\gamma_1\gamma_2\gamma_{3]} = \frac{i}{4!}\varepsilon_{0123}\,\varepsilon^{\mu\nu\rho\sigma}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma = \frac{i}{4!}\varepsilon^{\mu\nu\rho\sigma}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma$.

Thus: $\varepsilon_{\mu\nu\alpha\beta}\,tr(\gamma_5\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta) = 4i\,\varepsilon_{\mu\nu\alpha\beta}\varepsilon^{\mu\nu\alpha\beta}$

$\rightarrow$ $tr[(\gamma_5)^2] = 4$ $\rightarrow$ $tr(1_4) = 4$, which is identically satisfied.

The Dirac field

Part A.

① Assume that $[\psi] = [\bar\psi] = [M]^{\alpha}$.

Since $[d^4x] = [M]^{-4}$, $[\partial] = [M]^1$,
for the action S to be dimensionless we should require that $\alpha = \frac{3}{2} \rightarrow [\psi] = [\bar\psi] = [M]^{3/2}$.

② We start from

$$S = \int d^4x \left[i\bar\psi\gamma^\mu\partial_\mu\psi - m\bar\psi\psi \right]$$

$$\rightarrow S^\dagger = \int d^4x \left[(i\bar\psi\gamma^\mu\partial_\mu\psi)^\dagger - (m\bar\psi\psi)^\dagger \right]$$

$$= \int d^4x \left[-i\,\partial_\mu\psi^\dagger\gamma^{\mu\dagger}\bar\psi^\dagger - m\,\psi^\dagger\bar\psi^\dagger \right]$$

$$= \int d^4x \left[-i\,\partial_\mu \underbrace{\psi^\dagger\gamma^0}_{\bar\psi}\gamma^\mu\gamma^0(\psi^\dagger\gamma^0)^\dagger - m\,\psi^\dagger(\psi^\dagger\gamma^0)^\dagger \right]$$

$$= \int d^4x \left[-i\,\partial_\mu\bar\psi\gamma^\mu\psi - m\bar\psi\psi \right], \quad (*)$$

where we used $\gamma^{0\dagger} = \gamma^0$, $(\gamma^0)^2 = 1$.

where we used $\gamma^{0\dagger} = \gamma^0, (\gamma^0)^2 = 1$.

Notice that

$$\partial_\mu(\bar{\psi}\gamma^\mu\psi) = \partial_\mu\bar{\psi}\gamma^\mu\psi + \bar{\psi}\gamma^\mu\partial_\mu\psi$$

$$\rightarrow \partial_\mu\bar{\psi}\gamma^\mu\psi = -\bar{\psi}\gamma^\mu\partial_\mu\psi + \partial_\mu(\bar{\psi}\gamma^\mu\psi)$$

Plugging the above into ⊛, we observe

$$S^\dagger = S - i\underbrace{\int d^4x\, \partial_\mu(\bar{\psi}\gamma^\mu\psi)}_{\text{total derivative}}$$

③ To find the equations of motion for ψ, we vary the action w.r.t. $\bar{\psi}$

$$\delta_{\bar{\psi}} S = \int d^4x\, \delta\bar{\psi}\,[i\gamma^\mu\partial_\mu\psi - m\psi] = 0$$

$$\boxed{\rightarrow i\gamma^\mu\partial_\mu\psi - m\psi = 0} \quad (*)$$

Similarly, $\delta_\psi S = \int d^4x\, [i\bar{\psi}\gamma^\mu\partial_\mu\delta\psi - m\bar{\psi}\delta\psi]$

$$= \int d^4x\, [-i\partial_\mu\bar{\psi}\gamma^\mu - m\bar{\psi}]\delta\psi = 0$$

$$\rightarrow \boxed{i\partial_\mu\bar{\psi}\gamma^\mu + m\bar{\psi} = 0} \quad (**)$$

④ $\psi \rightarrow \psi' = e^{i\alpha}\psi \Rightarrow \bar{\psi} \rightarrow \bar{\psi}' = \bar{\psi}e^{-i\alpha}$

It is easy to see that the action is invariant under this transformation.

Infinitesimally,

$\delta\psi = i\alpha\psi$, $\delta\bar{\psi} = -i\alpha\bar{\psi}$

$$\to j^\mu = \frac{\delta \mathcal{L}}{\delta \partial_\mu \psi}\delta\psi + \delta\bar{\psi}\cancel{\frac{\delta \mathcal{L}}{\delta \partial_\mu \bar{\psi}}}^{\,0} = -\bar{\psi}\gamma^\mu\psi$$

Therefore $\partial_\mu j^\mu = -\partial_\mu\bar{\psi}\gamma^\mu\psi - \bar{\psi}\gamma^\mu\partial_\mu\psi$

Using $\circledast$, $\circledast\circledast$ from the previous point, we observe that $\partial_\mu j^\mu = 0$.

⑤ We know that

$$T^{\mu\nu} = \frac{\delta \mathcal{L}}{\delta \partial_\mu \psi}\partial^\nu\psi + \partial^\nu\bar{\psi}\cancel{\frac{\delta \mathcal{L}}{\delta \partial_\mu \bar{\psi}}}^{\,0} - \eta^{\mu\nu}\mathcal{L}$$

$$= i\bar{\psi}\gamma^\mu\partial^\nu\psi - \eta^{\mu\nu}\mathcal{L}$$

$\to \partial_\mu T^{\mu\nu} = i\partial_\mu(\bar{\psi}\gamma^\mu\partial^\nu\psi) - \partial^\nu\mathcal{L} = 0$, on the equations of motion ($\mathcal{L} = 0$ too!)

⑥ $$\pi = \frac{\delta\mathcal{L}}{\delta\dot{\psi}_0} = i\bar{\psi}\gamma^0 = i\psi^+$$

$$\bar{\pi} = \frac{\delta\mathcal{L}}{\delta\dot{\bar{\psi}}_0} = 0$$

⑦ $$H = \int d^3\vec{x}\, T^{00} = \int d^3\vec{x}\,(i\bar{\psi}\gamma^0\dot{\psi} - \mathcal{L})$$

$$= \int d^3\vec{x}\,(-i\bar{\psi}\gamma^i\partial_i\psi + m\bar{\psi}\psi)$$

$$P^i = \int d^3\vec{x}\, T^{0i} = \int d^3\vec{x}\, (i\bar{\psi}\gamma^0\partial^i\psi)$$

$$= \int d^3\vec{x}\ i\psi^\dagger \partial^i\psi = \int d^3\vec{x}\, \pi\, \partial^i\psi$$