Quantum Field Theory (Quantum Electrodynamics)

Problem Set 5 20 & 22 November 2023

1. The Poincaré group

The aim of this exercise is to learn how the *Poincaré group* $\mathcal{P}$ acts on scalar fields $\phi(x)$. $\mathcal{P}$ contains the Lorentz group together with spacetime translations. $\mathcal{P}$ acts on a vector Φ_a ($a = 1, \dots, N$) as

$$\Phi_a \to \Gamma \cdot \Phi_a = D_N[\Gamma]_{ab}\, \Phi_b,$$

where $\Gamma \in \mathcal{P}$ and $D_N[\Gamma]_{ab}$ is an $N \times N$ matrix called *N-dimensional representation* of Γ. In the case of vector fields $\Phi_a(x)$, which are also functions of spacetime, the Poincaré group acts as

$$\Phi_a(x) \to \Gamma \cdot \Phi_a(x) = D_N[\Gamma]_{ab}\, \Phi_b(\Gamma^{-1}x) \equiv D_N[\Gamma]_{ab} D_\infty[\Gamma]\, \Phi_b(x),$$

where $D_N[\Gamma]_{ab}$ is the same matrix as before and $D_\infty[\Gamma]$ is called *field representation*. The joint representation $D_N[\Gamma]_{ab} D_\infty[\Gamma]$ is called *vector field representation*.
Since $\mathcal{P}$ is a Lie group, one can write any element $\Gamma \in \mathcal{P}$ as

$$\Gamma = \exp\left[-ia^\mu P_\mu + \frac{i}{2}\omega^{\mu\nu} M_{\mu\nu}\right],$$

where $M_{\mu\nu} = -M_{\nu\mu}$ are the generators of the Lorentz transformations and P_μ are the generators of translations. The generators satisfy the Poincaré algebra :

$$\begin{aligned}
&[M_{\mu\nu}, M_{\rho\sigma}] = i\eta_{\mu\rho} M_{\nu\sigma} + i\eta_{\nu\sigma} M_{\mu\rho} - i\eta_{\mu\sigma} M_{\nu\rho} - i\eta_{\nu\rho} M_{\mu\sigma},\\
&[M_{\mu\nu}, P_\rho] = i\eta_{\mu\rho} P_\nu - i\eta_{\nu\rho} P_\mu,\\
&[P_\mu, P_\nu] = 0.
\end{aligned}$$

An infinitesimal Poincaré transformation of a vector x^μ is given by

$$D_4[\Gamma]^\mu{}_\nu x^\nu \sim x^\mu + a^\mu + \omega^\mu{}_\nu x^\nu.$$

1. Argue why $\mathcal{P}$ acts on Lorentz scalars (elements of a one-dimensional vector space, not fields !) in the *trivial representation*, such that $P_\mu = M_{\mu\nu} = 0$. Do the generators satisfy the Poincaré algebra ?
2. Show that in the *scalar field representation* the generators are given by

$$P_\mu = -i\partial_\mu \qquad \text{and} \qquad M_{\mu\nu} = -i\left(x_\mu \partial_\nu - x_\nu \partial_\mu\right) .$$

   Hint : Use that $\Gamma \cdot \phi(x) = \phi((\Gamma^{-1})^\mu{}_\nu x^\nu)$ and Taylor expand both sides. For simplicity, consider the Lorentz transformations and translations separately.
3. Verify that the generators in the scalar field representation satisfy the Poincaré algebra. Is the scalar field representation unitary ?

2. Complex scalar field

Part A.

Let us consider a free massive complex scalar field. The action reads

$$S = \int \mathrm{d}^4x \left[\partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi\right] \ .$$

1. Introduce a pair of real fields ϕ_1 and ϕ_2, such that

   $$\phi = \frac{\phi_1 + i\phi_2}{\sqrt{2}} \ ,$$

   and rewrite the action in terms of them.
2. a) Check that the action is invariant under

   $$\phi \to \phi' = e^{i\alpha}\phi \ ,$$

   with α a real constant.

   b) Find the corresponding Noether current j^μ.

   c) Verify that it is conserved on the equations of motion.
3. The mode decomposition of the field operator reads

   $$\hat{\phi}(x) = \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3 2\omega_{\vec{p}}} \left[\hat{a}(\vec{p})e^{-ip\cdot x} + \hat{b}(\vec{p})^+ e^{ip\cdot x}\right] \ .$$

   Find $\hat{\phi}^*(x)$ as well as the conjugate momenta $\hat{\pi}(x)$ and $\hat{\pi}^*(x)$.
4. Compute the commutation relations for the ladder operators.
5. Express the charge operator

   $$\hat{Q} = \int \mathrm{d}^3\vec{x}\, \hat{j}^0 \ ,$$

   in terms of $\hat{a}, \hat{b}$ (and their conjugates).

Part B.

Consider now the following action in a d-dimensional spacetime

$$S = \int \mathrm{d}^d x \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi - igA^\mu\left(\phi\partial_\mu\phi^* - \phi^*\partial_\mu\phi\right) + \tilde{g}A_\mu A^\mu \phi^*\phi\right) \ ,$$

As in Part A ϕ is a complex scalar, while A_μ is the Maxwell field and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$; $m, g, \tilde{g}$ are constants.

Find the dimensions of $A_\mu, \phi, m, g, \tilde{g}$.