$$S = \int d^4x \left[\frac{1}{2}(\partial_\mu \varphi)^2 - \frac{m^2}{2}\varphi^2 \right]$$

$$\varphi(t,\vec{x}) = \int \frac{d^3\vec{p}}{(2\pi)^3\, 2\omega_{\vec{p}}} \left[a(\vec{p})\, e^{i(\vec{p}\cdot\vec{x} - \omega_{\vec{p}} t)} + a^*(-\vec{p})\, e^{i(\vec{p}\cdot\vec{x} + \omega_{\vec{p}} t)} \right]$$

(1) $$\pi(t,\vec{x}) \equiv \frac{\delta \mathcal{L}}{\delta(\partial_0 \varphi)} = \partial_0 \varphi = \dot{\varphi}(t,\vec{x})$$

$$\Rightarrow \pi(t,\vec{x}) = \partial_t \varphi(t,\vec{x}) = -\frac{i}{2} \int \frac{d^3\vec{p}}{(2\pi)^3} \left[a(\vec{p})\, e^{i(\vec{p}\cdot\vec{x} - \omega_{\vec{p}} t)} - a^*(-\vec{p})\, e^{i(\vec{p}\cdot\vec{x} + \omega_{\vec{p}} t)} \right]$$

(2) we promote $a(\vec{p}), a^*(\vec{p})$ to $\hat{a}(\vec{p}), \hat{a}^\dagger(\vec{p})$: this means we are promoting $\varphi(t,\vec{x}), \pi(t,\vec{x})$ to the operators $\hat{\varphi}(t,\vec{x}), \hat{\pi}(t,\vec{x})$. We require $\hat{\varphi}(t,\vec{x})$ and $\hat{\pi}(t,\vec{x})$ to satisfy the equal time canonical commutation relations that read:

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$$\boxed{\begin{aligned} &[\hat{\varphi}(t,\vec{x}),\hat{\varphi}(t,\vec{x}')]=0\\ &[\hat{\pi}(t,\vec{x}),\hat{\pi}(t,\vec{x}')]=0\\ &[\hat{\varphi}(t,\vec{x}),\hat{\pi}(t,\vec{x}')]=i\delta^{(3)}(\vec{x}-\vec{x}')\end{aligned}} \qquad \boxed{1}$$

Reminder: $\int d^3\vec{x}\, e^{-i\vec{p}'\cdot\vec{x}} e^{i\vec{p}\cdot\vec{x}} = (2\pi)^3\delta^{(3)}(\vec{p}-\vec{p}')$

$$\Rightarrow \int d^3\vec{x}\, e^{-i\vec{p}'\cdot\vec{x}}\hat{\varphi}(t,\vec{x}) = \int\frac{d^3\vec{p}}{(2\pi)^3 2\omega(\vec{p})}\Big[\hat{a}(\vec{p})e^{-i\omega_{\vec{p}}t}\underbrace{\int d^3\vec{x}\, e^{-i\vec{p}'\cdot\vec{x}}e^{i\vec{p}\cdot\vec{x}}}_{=(2\pi)^3\delta^{(3)}(\vec{p}-\vec{p}')} +$$

$$+\hat{a}^\dagger(-\vec{p})e^{i\omega_{\vec{p}}t}\int d^3\vec{x}\, e^{-i\vec{p}'\cdot\vec{x}}e^{i\vec{p}\cdot\vec{x}}\Big] =$$

$$=\frac{1}{2\omega_{\vec{p}'}}\left[\hat{a}(\vec{p}')e^{-i\omega_{\vec{p}'}t}+\hat{a}^\dagger(-\vec{p}')e^{i\omega_{\vec{p}'}t}\right] \qquad \boxed{2}$$

and $\int d^3\vec{x}\, e^{-i\vec{p}'\cdot\vec{x}}\hat{\pi}(t,\vec{x}) = -\frac{i}{2}\left[\hat{a}(\vec{p}')e^{-i\omega_{\vec{p}'}t}-\hat{a}^\dagger(-\vec{p}')e^{i\omega_{\vec{p}'}t}\right]$ $\boxed{3}$

$$\Rightarrow \boxed{2}\cdot\omega_{\vec{p}'}+\boxed{3}\cdot i = \frac{1}{2}\left(\hat{a}(\vec{p}')e^{-i\omega_{\vec{p}'}t}+\hat{a}^\dagger(-\vec{p}')e^{i\omega_{\vec{p}'}t}\right)+$$

$$+\frac{1}{2}\left(\hat{a}(\vec{p}')e^{-i\omega_{\vec{p}'}t}-\hat{a}^\dagger(-\vec{p}')e^{i\omega_{\vec{p}'}t}\right)=\hat{a}(\vec{p}')e^{-i\omega_{\vec{p}'}t}$$

$$\Leftrightarrow \boxed{\hat{a}(\vec{p})=\int d^3\vec{x}\, e^{-i\vec{p}\cdot\vec{x}}e^{i\omega_{\vec{p}}t}\left(\omega_{\vec{p}}\hat{\varphi}(t,\vec{x})+i\hat{\pi}(t,\vec{x})\right)} \qquad \boxed{4}$$

and $\boxed{2}\cdot\omega_{\vec{p}'}-\boxed{3}\cdot i = \frac{1}{2}\left(\hat{a}(\vec{p}')e^{-i\omega_{\vec{p}'}t}+\hat{a}^\dagger(-\vec{p}')e^{i\omega_{\vec{p}'}t}\right)-$

$$-\frac{1}{2}\left(\hat{a}(\vec{p}')e^{-i\omega_{\vec{p}'}t}-\hat{a}^\dagger(-\vec{p}')e^{i\omega_{\vec{p}'}t}\right)=\hat{a}^\dagger(-\vec{p}')e^{i\omega_{\vec{p}'}t}$$

$$\Leftrightarrow \boxed{\hat{a}^\dagger(\vec{p})=\int d^3\vec{x}\, e^{i\vec{p}\cdot\vec{x}}e^{-i\omega_{\vec{p}}t}\left(\omega_{\vec{p}}\hat{\varphi}(t,\vec{x})-i\hat{\pi}(t,\vec{x})\right)} \qquad \boxed{5}$$

consistency check: $(\hat{a}(\vec{p}))^\dagger=\hat{a}^\dagger(\vec{p})$ ✓ because $\hat{\varphi},\hat{\pi}$ are real

• $\boxed{[\hat{a}(\vec{p}),\hat{a}(\vec{p}')]}=\Big[\int d^3\vec{x}\, e^{-i\vec{p}\cdot\vec{x}}e^{i\omega_{\vec{p}}t}\left(\omega_{\vec{p}}\hat{\varphi}(t,\vec{x})+i\hat{\pi}(t,\vec{x})\right),$

$$\int d^3\vec{x}'\, e^{-i\vec{p}'\cdot\vec{x}'}e^{i\omega_{\vec{p}'}t}\left(\omega_{\vec{p}'}\hat{\varphi}(t,\vec{x}')+i\hat{\pi}(t,\vec{x}')\right)\Big]=$$

$$=\int d^3\vec{x}\int d^3\vec{x}'\, e^{-i\vec{p}\cdot\vec{x}}e^{i\omega_{\vec{p}}t}e^{-i\vec{p}'\cdot\vec{x}'}e^{i\omega_{\vec{p}'}t}\Big(\omega_{\vec{p}}\omega_{\vec{p}'}\overbrace{[\hat{\varphi}(t,\vec{x}),\hat{\varphi}(t,\vec{x}')]}^{=0}+$$

$$+i\omega_{\vec{p}}[\hat{\varphi}(t,\vec{x}),\hat{\pi}(t,\vec{x}')]+i\omega_{\vec{p}'}[\hat{\pi}(t,\vec{x}),\hat{\varphi}(t,\vec{x}')]-\underbrace{[\hat{\pi}(t,\vec{x}),\hat{\pi}(t,\vec{x}')]}_{=0}\Big)=$$

$$=\int d^3\vec{x}\int d^3\vec{x}'\, e^{-i\vec{p}\cdot\vec{x}}e^{i\omega_{\vec{p}}t}e^{-i\vec{p}'\cdot\vec{x}'}e^{i\omega_{\vec{p}'}t}\Big(-\omega_{\vec{p}}\delta^{(3)}(\vec{x}-\vec{x}')+$$

$$+\omega_{\vec{p}'}\delta^{(3)}(\vec{x}-\vec{x}')\Big)=$$

$$=\underbrace{\int d^3\vec{x}\, e^{-i(\vec{p}+\vec{p}')\cdot\vec{x}}}_{=(2\pi)^3\delta^{(3)}(\vec{p}+\vec{p}')}e^{i(\omega_{\vec{p}}+\omega_{\vec{p}'})t}\left(\omega_{\vec{p}'}-\omega_{\vec{p}}\right)=\boxed{0}$$

i.e. gives $\vec{p}'=-\vec{p}$

reminder: $\omega_{\vec{p}}=\sqrt{\vec{p}^2+m^2}$ ⇝ i.e. $\omega_{\vec{p}}=\omega_{\vec{p}'}$

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- $\boxed{\left[\hat{a}^{\dagger}(\vec{p}), \hat{a}^{\dagger}(\vec{p}\,')\right]} = \underbrace{\hat{a}^{\dagger}(\vec{p})\,\hat{a}^{\dagger}(\vec{p}\,')}_{=\left(\hat{a}(\vec{p}\,')\,\hat{a}(\vec{p})\right)^{\dagger}} - \underbrace{\hat{a}^{\dagger}(\vec{p}\,')\,\hat{a}^{\dagger}(\vec{p})}_{=\left(\hat{a}(\vec{p})\,\hat{a}(\vec{p}\,')\right)^{\dagger}} = \Big(\underbrace{\left[\hat{a}(\vec{p}\,'), \hat{a}(\vec{p})\right]}_{=0}\Big)^{\dagger} = \boxed{0}$

- $\boxed{\left[\hat{a}(\vec{p}), \hat{a}^{\dagger}(\vec{p}\,')\right]} = \int d^3\vec{x} \int d^3\vec{x}\,'\, e^{-i\vec{p}\cdot\vec{x}} e^{i\omega_{\vec{p}} t} e^{i\vec{p}\,'\cdot\vec{x}\,'} e^{-i\omega_{\vec{p}\,'} t} \cdot$
  $\cdot \underbrace{\left(\left[\omega_{\vec{p}}\,\hat{\varphi}(t,\vec{x}) + i\hat{\pi}(t,\vec{x}),\ \omega_{\vec{p}\,'}\,\hat{\varphi}(t,\vec{x}\,') - i\hat{\pi}(t,\vec{x}\,')\right]\right)}_{} =$

$$= \omega_{\vec{p}}\,\omega_{\vec{p}\,'} \underbrace{\left[\hat{\varphi}(t,\vec{x}), \hat{\varphi}(t,\vec{x}\,')\right]}_{=0} - i\omega_{\vec{p}} \underbrace{\left[\hat{\varphi}(t,\vec{x}), \hat{\pi}(t,\vec{x}\,')\right]}_{=i\delta^{(3)}(\vec{x}-\vec{x}\,')} +$$

$$+ i\omega_{\vec{p}\,'} \underbrace{\left[\hat{\pi}(t,\vec{x}), \hat{\varphi}(t,\vec{x}\,')\right]}_{=-i\delta^{(3)}(\vec{x}-\vec{x}\,')} + \underbrace{\left[\hat{\pi}(t,\vec{x}), \hat{\pi}(t,\vec{x}\,')\right]}_{=0}$$

$$= \int d^3\vec{x} \int d^3\vec{x}\,'\, e^{i(\vec{p}\,'\cdot\vec{x}\,' - \vec{p}\cdot\vec{x})} e^{i(\omega_{\vec{p}} - \omega_{\vec{p}\,'})t} (\omega_{\vec{p}} + \omega_{\vec{p}\,'})\, \delta^{(3)}(\vec{x}-\vec{x}\,') =$$

$$= \int d^3\vec{x}\, e^{i(\vec{p}\,'-\vec{p})\cdot\vec{x}} e^{i(\omega_{\vec{p}} - \omega_{\vec{p}\,'})t} (\omega_{\vec{p}} + \omega_{\vec{p}\,'}) =$$

$$= (2\pi)^3 \delta^{(3)}(\vec{p}-\vec{p}\,') e^{i(\omega_{\vec{p}} - \omega_{\vec{p}\,'})t} (\omega_{\vec{p}} + \omega_{\vec{p}\,'}) = \qquad \omega_{\vec{p}} = \sqrt{\vec{p}^{\,2} + m^2}$$

$$= \boxed{2\omega_{\vec{p}} (2\pi)^3 \delta^{(3)}(\vec{p}-\vec{p}\,')}$$

(3) Energy-momentum tensor

[Peskin, Schroeder: p. 18-19]

consider an infinitesimal spacetime translation:
$x^{\mu} \to x'^{\mu} = x^{\mu} - \delta x^{\mu} \equiv x^{\mu} - a^{\mu}$; we can alternatively describe this as

$$\varphi(x) \to \varphi'(x) = \varphi(x+a) \approx \varphi(x) + a^{\mu}\partial_{\mu}\varphi(x) \equiv \varphi + \delta\varphi \qquad \delta\varphi = (\partial^{\nu}\varphi)\, a_{\nu}$$

'S2 E1.(a)1.: Euler-Lagrange eqns.: $\dfrac{\delta\mathcal{L}}{\delta\varphi} - \partial_{\mu}\left(\dfrac{\delta\mathcal{L}}{\delta(\partial_{\mu}\varphi)}\right) = 0 \qquad \boxed{1}$

$\mathcal{L}$ is also a scalar & transforms as:

$$\mathcal{L} \to \mathcal{L}' = \mathcal{L} + \delta\mathcal{L} \quad w/ \quad \delta\mathcal{L} = \frac{\delta\mathcal{L}}{\delta x^{\mu}}\delta x^{\mu} = (\partial_{\mu}\mathcal{L})\, a^{\mu} = \partial_{\mu}(\eta^{\mu\nu}\mathcal{L})\, a_{\nu} \qquad \boxed{2}$$

we can also vary $\mathcal{L}$ as:

$$\delta\mathcal{L} = \frac{\delta\mathcal{L}}{\delta\varphi}\delta\varphi + \frac{\delta\mathcal{L}}{\delta(\partial_{\mu}\varphi)} \underbrace{\delta(\partial_{\mu}\varphi)}_{=\partial_{\mu}(\delta\varphi)} \overset{\boxed{1}}{=} \left[\partial_{\mu}\left(\frac{\delta\mathcal{L}}{\delta(\partial_{\mu}\varphi)}\right)\right]\delta\varphi + \frac{\delta\mathcal{L}}{\delta(\partial_{\mu}\varphi)}\partial_{\mu}(\delta\varphi) =$$

$$= \partial_{\mu}\left(\frac{\delta\mathcal{L}}{\delta(\partial_{\mu}\varphi)}\delta\varphi\right) = \partial_{\mu}\left(\frac{\delta\mathcal{L}}{\delta(\partial_{\mu}\varphi)}(\partial^{\nu}\varphi)\right) a_{\nu} \qquad \boxed{3}$$

$$\boxed{2} = \boxed{3} \iff \partial_{\mu}\Big(\underbrace{\frac{\delta\mathcal{L}}{\delta(\partial_{\mu}\varphi)}(\partial^{\nu}\varphi) - \eta^{\mu\nu}\mathcal{L}}_{\equiv T^{\mu\nu}}\Big) = 0 \implies 4 \text{ separately conserved currents!}$$

e.g.: time transl. inv. gives cons. of energy

$\Rightarrow$ Energy-momentum tensor $\boxed{T^{\mu\nu} = \dfrac{\delta\mathcal{L}}{\delta(\partial_{\mu}\varphi)}(\partial^{\nu}\varphi) - \eta^{\mu\nu}\mathcal{L}}$

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$$\hat{T}^{\mu\nu} = (\partial^\mu \hat{\varphi})(\partial^\nu \hat{\varphi}) - \frac{\eta^{\mu\nu}}{2}\left[(\partial_\lambda \hat{\varphi})^2 - m^2 \hat{\varphi}^2\right] \quad ; \quad \hat{P}^\mu = \int d^3\vec{x}\, \hat{T}^{0\mu} \qquad \boxed{4}$$

$$\Rightarrow \hat{T}^{00} = \underbrace{(\partial^0 \hat{\varphi})^2}_{=\hat{\pi}^2} - \frac{1}{2}\left[\hat{\pi}^2 - (\partial_i \hat{\varphi})^2 - m^2\hat{\varphi}^2\right] = \frac{1}{2}\left[\hat{\pi}^2 + (\partial_i \hat{\varphi})^2 + m^2 \hat{\varphi}^2\right] \equiv \hat{\mathcal{H}}$$

$x^\mu = (x^0, x^1, x^2, x^3)^T$ $\quad \partial^\mu = \frac{\partial}{\partial x_\mu} \Rightarrow \partial^i = \frac{\partial}{\partial x_i} = -\partial_i$

$x_\mu = (x^0, -x^1, -x^2, -x^3)^T$

cf. PS2 E1.(a) 5.

the conserved charge associated w/ time translations is the Hamiltonian $\hat{H} = \hat{P}^0$; the conserved charges associated w/ spatial translations are $\hat{P}^i$: we interpret this as the (physical) momentum carried by the field (not to be confused w/ the canonical momentum).

Instead of quantizing in the <u>Heisenberg</u> picture, i.e. time-dependent operators and equal-time canonical commutation relations along with time-independent states, we could equally well quantize in the <u>Schrödinger</u> picture: time-independent operators, still „equal-time" commutation relations and time-dependent states.

So, in the Schrödinger picture we have: (i.e. dropping t-dep. of φ & π)

$$\boxed{\begin{aligned} \hat{\varphi}(\vec{x}) &= \int \frac{d^3\vec{p}}{(2\pi)^3 2\omega_{\vec{p}}} \left[\hat{a}(\vec{p}) + \hat{a}^\dagger(-\vec{p})\right] e^{i\vec{p}\cdot\vec{x}} \\ \hat{\pi}(\vec{x}) &= \int \frac{d^3\vec{p}}{(2\pi)^3 2\omega_{\vec{p}}} (-i\omega_{\vec{p}}) \left[\hat{a}(\vec{p}) - \hat{a}^\dagger(-\vec{p})\right] e^{i\vec{p}\cdot\vec{x}} \end{aligned}}$$

$$\int d^3\vec{x}\, e^{i(\vec{p}+\vec{p}\,')\cdot\vec{x}} = (2\pi)^3 \delta^{(3)}(\vec{p}+\vec{p}\,')$$

$$\Rightarrow \hat{P}^0 = \int d^3\vec{x}\, \hat{\mathcal{H}} = \frac{1}{2}\int d^3\vec{x} \frac{d^3\vec{p}}{(2\pi)^3 2\omega_{\vec{p}}} \frac{d^3\vec{p}\,'}{(2\pi)^3 2\omega_{\vec{p}\,'}} \Big\{ -\omega_{\vec{p}}\omega_{\vec{p}\,'} \left(\hat{a}(\vec{p}) - \hat{a}^\dagger(-\vec{p})\right)\left(\hat{a}(\vec{p}\,') - \hat{a}^\dagger(-\vec{p}\,')\right) +$$

$$+ (\vec{p}\cdot\vec{p}\,' + m^2)\left(\hat{a}(\vec{p}) + \hat{a}^\dagger(-\vec{p})\right)\left(\hat{a}(\vec{p}\,') + \hat{a}^\dagger(-\vec{p}\,')\right)\Big\} e^{i(\vec{p}+\vec{p}\,')\cdot\vec{x}} \overset{\int d^3\vec{x},\ \text{kill } \int d^3\vec{p}\,' \text{ w/ } \delta^{(3)}(\vec{p}+\vec{p}\,')}{=}$$

$$= \frac{1}{2}\int \frac{d^3\vec{p}}{(2\pi)^3 2\omega_{\vec{p}}} \frac{\omega_{\vec{p}}^2}{2\omega_{\vec{p}}} \Big\{ \underline{-\hat{a}(\vec{p})\hat{a}(-\vec{p})} + \hat{a}(\vec{p})\hat{a}^\dagger(\vec{p}) +$$

$$+ \hat{a}^\dagger(-\vec{p})\hat{a}(-\vec{p}) \underline{- \hat{a}^\dagger(-\vec{p})\hat{a}^\dagger(\vec{p})} + \underline{\hat{a}(\vec{p})\hat{a}(-\vec{p})} + \hat{a}(\vec{p})\hat{a}^\dagger(\vec{p}) +$$

$$+ \hat{a}^\dagger(-\vec{p})\hat{a}(-\vec{p}) + \underline{\hat{a}^\dagger(-\vec{p})\hat{a}^\dagger(\vec{p})} \Big\} =$$

$$= \frac{1}{2}\int \frac{d^3\vec{p}}{(2\pi)^3 2\omega_{\vec{p}}} \omega_{\vec{p}} \Big(\underbrace{\hat{a}^\dagger(-\vec{p})\hat{a}(-\vec{p})}_{\substack{\text{change int. var.} \\ \vec{p}\to-\vec{p}}} + \hat{a}(\vec{p})\hat{a}^\dagger(\vec{p}) \Big) = \frac{1}{2}\int \frac{d^3\vec{p}}{(2\pi)^3 2\omega_{\vec{p}}} \omega_{\vec{p}} \left(\hat{a}^\dagger(\vec{p})\hat{a}(\vec{p}) + \hat{a}(\vec{p})\hat{a}^\dagger(p)\right)$$

Note that $\hat{a}(\vec{p})\hat{a}^\dagger(\vec{p}) = \hat{a}^\dagger(\vec{p})\hat{a}(\vec{p}) + \underbrace{\left(\hat{a}(\vec{p})\hat{a}^\dagger(\vec{p}) - \hat{a}^\dagger(\vec{p})\hat{a}(\vec{p})\right)}_{= [\hat{a}(\vec{p}), \hat{a}^\dagger(\vec{p})] = 2\omega_{\vec{p}}(2\pi)^3 \delta^{(3)}(\vec{0})}$

Then $$\boxed{\hat{P}^0 = \int \frac{d^3\vec{p}}{(2\pi)^3 2\omega_{\vec{p}}} \omega_{\vec{p}} \hat{a}^\dagger(\vec{p})\hat{a}(\vec{p}) + \int \frac{d^3\vec{p}}{(2\pi)^3}\frac{\omega_{\vec{p}}}{2}\underbrace{\delta^{(3)}(\vec{0})}_{\text{this is infinite!}}}$$

Note from $\int d^3\vec{x}\, e^{i\vec{p}\cdot\vec{x}} = (2\pi)^3\delta^{(3)}(\vec{p})$,
that $(2\pi)^3\delta^{(3)}(\vec{0}) = \int d^3\vec{x} = V$, the total volume.
The infinite constant term is thus an infinite (but const.) energy shift that contributes to the total energy of the system. It is simply the sum over all modes of the zero-point energies $\frac{\omega_{\vec{p}}}{2}$. This is the vacuum energy $E_{vac} = \frac{V}{(2\pi)^3}\int\frac{d^3\vec{p}}{(2\pi)^3}\frac{\omega_{\vec{p}}}{2}$ and is given by the sum of energies of zero-modes of each quantum oscillator. Remember, we treated the Fourier modes of the field $\hat{\varphi}$ as each being an independent oscillator with its own $\hat{a}$ & $\hat{a}^\dagger$.
However, only energy <u>differences</u>, and not absolute energies are observed experimentally. We will therefore ignore the vacuum energy. Formally, this is done by performing

<u>Normal ordering.</u>

An operator $\hat{O}$ is normal-ordered if all creation/annihilation operators appear on the left/right. For such $\hat{O}$ we write $:\hat{O}:$

for example $:\hat{a}^\dagger(\vec{p})\hat{a}(\vec{p}) + \hat{a}(\vec{p})\hat{a}^\dagger(\vec{p}): \; = 2\hat{a}^\dagger(\vec{p})\hat{a}(\vec{p})$

clearly, $\langle 0|:\hat{O}:|0\rangle = 0$ $\quad \hat{a}_k|n_k\rangle = \sqrt{n_k}\,|n_k - 1\rangle$; $\hat{a}^\dagger_k|n_k\rangle = \sqrt{n_k+1}\,|n_k+1\rangle$

$\Rightarrow$ we redefine $\hat{H}$ by subtracting $\langle 0|\hat{H}|0\rangle = E_{vac}$; this removes the infinite contribution so that we now have $\langle 0|\hat{H}|0\rangle = 0$.

$k = (\omega_k, \vec{k})$

def. vac. $|0\rangle$ analogously: $a(k)|0\rangle = 0$, $a^\dagger(k)|0\rangle = |k\rangle$

We analogously redefine all other operators.

From $\boxed{4}$: $\hat{T}^{0i} = (\partial^0\hat{\varphi})(\partial^i\hat{\varphi}) = -\hat{\pi}(\partial_i\hat{\varphi})$

$$\Rightarrow \hat{P}^i = \int d^3\vec{x}\, \hat{T}^{0i} = \int d^3\vec{x} \frac{d^3\vec{p}}{(2\pi)^3 2\omega_{\vec{p}}}\frac{d^3\vec{p}'}{(2\pi)^3 2\omega_{\vec{p}'}}(-\omega_{\vec{p}}p'^i)(\hat{a}(\vec{p}) - \hat{a}^\dagger(-\vec{p}))(\hat{a}(\vec{p}') + \hat{a}^\dagger(-\vec{p}'))\cdot$$
$$\cdot e^{i(\vec{p}+\vec{p}')\cdot\vec{x}} = \int\frac{d^3\vec{p}}{(2\pi)^3 2\omega_{\vec{p}}}\frac{1}{2\omega_{\vec{p}}}(\omega_{\vec{p}}p^i)\Big(\hat{a}(\vec{p}) - \hat{a}^\dagger(-\vec{p})\Big)\Big(\hat{a}(-\vec{p}) + \hat{a}^\dagger(\vec{p})\Big) =$$

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$$= \int \frac{d^3\vec{p}}{(2\pi)^3 2\omega_{\vec{p}}} \frac{p^i}{2} \Big(\underbrace{\hat{a}(\vec{p})\hat{a}^\dagger(\vec{p})}_{= [\hat{a}(\vec{p}), \hat{a}^\dagger(\vec{p})] + \hat{a}^\dagger(\vec{p})\hat{a}(\vec{p})} - \hat{a}^\dagger(-\vec{p})\hat{a}(-\vec{p}) \Big) + \underbrace{\int \frac{d^3\vec{p}}{(2\pi)^3 2\omega_{\vec{p}}} \frac{p^i}{2} \Big(\hat{a}(\vec{p})\hat{a}(-\vec{p}) - \hat{a}^\dagger(-\vec{p})\hat{a}^\dagger(\vec{p}) \Big)}_{\text{int. var. change } \vec{p} \to -\vec{p}}$$

$$\underbrace{[\hat{a}(\vec{p}), \hat{a}^\dagger(\vec{p})]}_{= 2\omega_{\vec{p}} (2\pi)^3 \delta^{(3)}(\vec{0})}$$

2nd term of 1st integral:

$$\int_{-\infty}^{+\infty} \frac{d^3\vec{p}}{(2\pi)^3 2\omega_{\vec{p}}} \frac{p^i}{2} \Big(- \hat{a}^\dagger(-\vec{p})\hat{a}(-\vec{p}) \Big) \underset{\vec{p} \to -\vec{p}}{=}$$

$$= \int_{+\infty}^{-\infty} \frac{d^3(-\vec{p})}{(2\pi)^3 2\omega_{\vec{p}}} \frac{(-p^i)}{2} \Big(- \hat{a}^\dagger(\vec{p})\hat{a}(\vec{p}) \Big) =$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3 2\omega_{\vec{p}}} \frac{p^i}{2} \hat{a}^\dagger(\vec{p})\hat{a}(\vec{p})$$

$$= + \int \frac{d^3\vec{p}}{(2\pi)^3 2\omega_{\vec{p}}} \frac{(-p^i)}{2} \Big(\underbrace{\hat{a}(-\vec{p})\hat{a}(\vec{p})}_{= \hat{a}(\vec{p})\hat{a}(-\vec{p}) \text{ since } [\hat{a}(\vec{p}), \hat{a}(\vec{p}')] = 0} - \hat{a}^\dagger(\vec{p})\hat{a}^\dagger(-\vec{p}) \Big) =$$

$$= - \int \frac{d^3\vec{p}}{(2\pi)^3 2\omega_{\vec{p}}} \frac{p^i}{2} \Big(\hat{a}(\vec{p})\hat{a}(-\vec{p}) - \hat{a}^\dagger(-\vec{p})\hat{a}^\dagger(\vec{p}) \Big) = 0$$

$$\Rightarrow \boxed{\hat{P}^i \overset{N.O.}{=} \int \frac{d^3\vec{p}}{(2\pi)^3 2\omega_{\vec{p}}} p^i \hat{a}^\dagger(\vec{p})\hat{a}(\vec{p})}$$

(4) back to Heisenberg picture: $e^{i\hat{H}t} \hat{\varphi}(0, \vec{x}) e^{-i\hat{H}t}$

since $\hat{\varphi}$ is expressed in terms of $\hat{a}$ & $\hat{a}^\dagger$, we should compute

$$e^{i\hat{H}t} \hat{a}(\vec{p}) e^{-i\hat{H}t} \equiv \hat{a}(t, \vec{p})$$

differentiate both sides w.r.t. t:

$$i e^{i\hat{H}t} \underbrace{\Big(\hat{H}\hat{a}(\vec{p}) - \hat{a}(\vec{p})\hat{H} \Big)}_{= [\hat{H}, \hat{a}(\vec{p})]} e^{-i\hat{H}t} = \frac{d}{dt} \hat{a}(t, \vec{p})$$

$$[\hat{H}, \hat{a}(\vec{p})] = \int \frac{d^3\vec{p}\,'}{(2\pi)^3 2\omega_{\vec{p}'}} \omega_{\vec{p}'} \underbrace{\Big[\hat{a}^\dagger(\vec{p}\,')\hat{a}(\vec{p}\,'), \hat{a}(\vec{p}) \Big]}_{= \hat{a}^\dagger(\vec{p}\,') \underbrace{[\hat{a}(\vec{p}\,'), \hat{a}(\vec{p})]}_{=0} + \underbrace{[\hat{a}^\dagger(\vec{p}\,'), \hat{a}(\vec{p})]}_{= -2\omega_{\vec{p}} (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{p}\,')} \hat{a}(\vec{p}\,')} = -\omega_{\vec{p}} \hat{a}(\vec{p})$$

$$\Rightarrow \frac{d}{dt} \hat{a}(t, \vec{p}) = -i\omega_{\vec{p}} \hat{a}(t, \vec{p})$$

$(AB)^\dagger = B^\dagger A^\dagger$

$$\Rightarrow \boxed{\hat{a}(t, \vec{p}) = e^{-i\omega_{\vec{p}} t} \hat{a}(\vec{p})}$$

$$\& \ \boxed{\hat{a}^\dagger(t, \vec{p})} = \Big(e^{i\hat{H}t} \hat{a}(\vec{p}) e^{-i\hat{H}t} \Big)^\dagger = e^{i\hat{H}t} \hat{a}^\dagger(\vec{p}) e^{-i\hat{H}t} = \boxed{e^{i\omega_{\vec{p}} t} \hat{a}^\dagger(\vec{p})}$$

see p. 5

$$e^{i\hat{H}t}\hat{\varphi}(0,\vec{x})e^{-i\hat{H}t} = \int\frac{d^3\vec{p}}{(2\pi)^3 2\omega_{\vec{p}}}\left[e^{i\hat{H}t}\hat{a}(\vec{p})e^{-i\hat{H}t} + e^{i\hat{H}t}\hat{a}^\dagger(-\vec{p})e^{-i\hat{H}t}\right]e^{i\vec{p}\cdot\vec{x}} =$$

$$= \int\frac{d^3\vec{p}}{(2\pi)^3 2\omega_{\vec{p}}}\left[\hat{a}(\vec{p})e^{i(\vec{p}\cdot\vec{x}-\omega_{\vec{p}}t)} + \hat{a}^\dagger(-\vec{p})e^{i(\vec{p}\cdot\vec{x}+\omega_{\vec{p}}t)}\right] \equiv \hat{\varphi}(t,\vec{x})$$

We can thus interpret the Hamiltonian as the generator of time translations.

(5) We want to find $e^{i\hat{P}_i y^i}\hat{\varphi}(0,\vec{x})e^{-i\hat{P}_i y^i}$

We expect that this operation will now shift the coordinates (momentum is the generator of spatial translations). Similarly to what we did in the previous question, one can show that

$$e^{i\hat{P}_i y^i}\hat{\varphi}(0,\vec{x})e^{-i\hat{P}_i y^i} \equiv \hat{\varphi}(0,\vec{x}+\vec{y}).$$

(6) $$|\vec{p}_1,\vec{p}_2\rangle = \hat{a}^\dagger(\vec{p}_1)\hat{a}^\dagger(\vec{p}_2)|0\rangle \underset{[\hat{a}^\dagger(\vec{p}),\hat{a}^\dagger(\vec{p}\,')]=0}{=} \hat{a}^\dagger(\vec{p}_2)\hat{a}^\dagger(\vec{p}_1)|0\rangle = |\vec{p}_2,\vec{p}_1\rangle$$

this is symmetric under the interchange of $\vec{p}_1, \vec{p}_2$

$\Rightarrow$ Bose-Einstein statistics.

(7) we had
$$\hat{P}^0 = \int\frac{d^3\vec{p}}{(2\pi)^3 2\omega_{\vec{p}}}\,\omega_{\vec{p}}\,\hat{a}^\dagger(\vec{p})\hat{a}(\vec{p})$$
$$\hat{P}^i = \int\frac{d^3\vec{p}}{(2\pi)^3 2\omega_{\vec{p}}}\,p^i\,\hat{a}^\dagger(\vec{p})\hat{a}(\vec{p})$$

with $p^\mu = (\omega_{\vec{p}},\vec{p})$ we can define the four-momentum operator:

$$\boxed{\hat{P}^\mu = \int\frac{d^3\vec{p}}{(2\pi)^3 2\omega_{\vec{p}}}\,p^\mu\,\hat{a}^\dagger(\vec{p})\hat{a}(\vec{p})}$$

using the commutation relations of $\hat{a}, \hat{a}^\dagger$ and $\hat{a}(\vec{p})|0\rangle = 0$

we find $P^\mu|\vec{p}_1,\vec{p}_2\rangle = (p_1^\mu + p_2^\mu)|\vec{p}_1,\vec{p}_2\rangle$

$\Rightarrow |\vec{p}_1,\vec{p}_2\rangle$ is an eigenstate of the energy & momentum operators with eigenvalues $\omega = \omega_{\vec{p}_1}+\omega_{\vec{p}_2}$, $\vec{p} = \vec{p}_1+\vec{p}_2$, respectively.

(8) Particle number operator $\boxed{\hat{N} = \int\frac{d^3\vec{p}}{(2\pi)^3 2\omega_{\vec{p}}}\,\hat{a}^\dagger(\vec{p})\hat{a}(\vec{p})}$, cf. $\hat{P}^\mu$ in (7).

(9) classically, we obtain

$P^\mu = \int\frac{d^3\vec{p}}{(2\pi)^3 2\omega_{\vec{p}}}\,p^\mu\,a^*(\vec{p})a(\vec{p})$; quantizing, i.e. promoting $a, a^* \to \hat{a}, \hat{a}^\dagger$, gives $\hat{P}^\mu$ from (7), but this time without the trouble of having to introduce normal ordering.