# Quantum Field Theory (Quantum Electrodynamics)

**Problem Set 4** 13 & 15 November 2023

## 1. Canonical quantization of the real scalar field

We have already seen that the action of the (free) real massive scalar field in four spacetime dimensions is

$$S = \int \mathrm{d}^4x \left[\frac{1}{2}(\partial_\mu\phi)^2 - \frac{m^2}{2}\phi^2\right] \ .$$

In the problem set 2 we showed that the solution to the Klein-Gordon equation reads

$$\phi(\vec{x},t) = \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3 2\omega(\vec{p})} \left[a(\vec{p})e^{i\vec{p}\cdot\vec{x}-i\omega_{\vec{p}}t} + a(-\vec{p})^* e^{i\vec{p}\cdot\vec{x}+i\omega_{\vec{p}}t}\right] \ .$$

1. Find the canonical momentum $\pi(t,\vec{x})$ in terms of $a$ and $a^*$.
2. Promote $a$ and $a^*$ to the ladder operators $\hat{a}$ and $\hat{a}^+$. Requiring that the field and its conjugate momentum satisfy the equal time canonical commutation relations, derive the commutation relations for $\hat{a}$ and $\hat{a}^+$.
3. Starting from $\hat{P}^\mu = \int d^3x \hat{T}^{0\mu}$, find the exact form of the energy and momentum operators $\hat{P}^0$ and $\hat{P}^i$, respectively. Simplify the expressions by using the commutation relations from the previous point.

   *Hint : Remember that $P^0$ is the Hamiltonian, so you can use the expression from Problem Set 2, Exercise 1. You may get rid of divergent terms by redefining the vacuum energy.*
4. Compute $e^{i\hat{H}t}\hat{\phi}(0,\vec{x})e^{-i\hat{H}t}$, where $\hat{H}$ is the Hamiltonian of the system.
5. Compute $e^{i\hat{P}_i y^i}\hat{\phi}(0,\vec{x})e^{-i\hat{P}_i y^i}$, with $y^i$ a constant three-vector.

   *Hint : You may not do the computation, just argue qualitatively.*
6. Create a two-particles state $|\vec{p}_1,\vec{p}_2\rangle$. What statistics does this state obey ?
7. Show that $|\vec{p}_1,\vec{p}_2\rangle$ is an eigenstate of the energy and momentum operators. Find the corresponding eigenvalues.
8. Construct an operator that counts the number of particles in the states. Does this operator have any similarities with the energy and momentum ones ?
9. What will be different if we compute the energy and momentum classically in terms of $a$ and $a^*$ quantities and then replace them by the $\hat{a}$ and $\hat{a}^+$ operators ?