3. Liouville theory

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1. Let's assume that $\phi(x) \to \phi'(x) = \alpha^{\Delta}\phi(\alpha x)$

$$\to S' = \int d^2x \left[\frac{1}{2}\left(\frac{\partial}{\partial x^{\mu}}\phi'(x)\right)^2 - a e^{b\phi'(x)} \right]$$

$$= \int d^2x \left[\frac{1}{2}\alpha^{2(\Delta+1)}\left(\frac{\partial}{\partial(\alpha x^{\mu})}\phi(\alpha x)\right)^2 - a e^{\alpha^{\Delta} b\phi(\alpha x)} \right]$$

changing variables to $y^{\mu} = \alpha x^{\mu}$,

$$S' = \int d^2y \left[\frac{1}{2}\alpha^{2\Delta}(\partial_{\mu}\phi(y))^2 - a\alpha^{-2} e^{\alpha^{\Delta} b\phi(y)} \right]$$

We notice that the kinetic term is invariant provided that $\Delta = 0$.

$$\to S' = \int d^2y \left[\frac{1}{2}(\partial_{\mu}\phi(y))^2 - a\alpha^{-2} e^{b\phi(y)} \right]$$

However, the potential part is not invariant. But, what may save the day, is the fact that we have an exponential, so if say we shift ϕ by a constant, the change in the potential will be multipliative.

Indeed, let's assume that

$\phi \to \phi'' = \phi + c$, with $c =$ constant

$$\rightarrow S'' = \int d^2y \left[\frac{1}{2}(\partial_\mu \varphi(y))^2 - a\alpha^{-2} e^{b\phi(y)+bc}\right]$$ 10/10

$$= \int d^2y \left[\frac{1}{2}(\partial_\mu \varphi(y))^2 - a \underbrace{\alpha^{-2} e^{bc}}_{=1} e^{b\phi(y)}\right]$$

$$\rightarrow c = \frac{2\log\alpha}{b}$$

Therefore, "dilatations" in the two dimensional Liouville model correspond to:

$$\phi(x) \rightarrow \phi'(x) = \varphi(\alpha x) + \frac{2\log\alpha}{b}$$

2. For $\alpha = 1+\varepsilon$, $\varepsilon << 1$,

$$\delta\phi \approx \varepsilon\left(x^\mu \partial_\mu \phi + \frac{2}{b}\right) + \mathcal{O}(\varepsilon^2)$$

Thus, from Noether's theorem:

$$j_\mu = x^\nu \partial_\mu \phi \, \partial_\nu \phi + \frac{2}{b}\partial_\mu \phi - x_\mu \mathcal{L}$$

$$= x^\nu \partial_\mu \varphi \, \partial_\nu \varphi - x_\mu \frac{1}{2}(\partial_\nu \varphi)^2 + a x_\mu e^{b\phi}$$

$$+ \frac{2}{b}\partial_\mu \phi$$

$$\rightarrow \partial_\mu j^\mu = \ldots = \left(x^\mu \partial_\mu \phi + \frac{2}{b}\right)\underbrace{\left(\partial^2 \phi + a b e^{b\phi}\right)}_{\text{eq. of motion!}} = 0$$

1. Maxwell theory

① Dimensional analysis:

$[d^4x] = [M]^{-4}$, $[F] = [\partial][A] = [M]^1 [M]^\alpha$

$\to$ $[F^2] = [M]^{2+2\alpha}$. Thus for the action to be dimensionless, we should require that

$2 + 2\alpha - 4 = 0 \to \boxed{\alpha = 1} \to [A] = [M]^1$.

② We start from

$$S = \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + A_\mu j^\mu \right)$$

and vary with respect to the vector field:

$$\delta_A S = \int d^4x \left(- F^{\mu\nu} \partial_\mu \delta A_\nu + j^\mu \delta A_\mu \right) = 0 \ \forall A_\mu.$$

$$\to \boxed{\partial_\mu F^{\mu\nu} = -j^\nu}$$

③ We first notice that the kinetic term for A_μ, i.e. $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is manifestley invariant under $A_\mu \to A'_\mu = A_\mu + \partial_\mu \alpha$.

For the source term, we observe that

$$S \to S' = \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + A_\mu j^\mu + j^\mu \partial_\mu \alpha \right)$$

Integrating the last term by parts & dropping the boundary contribution, we see that for $S' = S \rightarrow \partial_\mu j^\mu = 0$, i.e. the source has to be conserved.

④ Since $F_{\mu\nu}$ is gauge-invariant object, so is its dual $\tilde{F}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}$.

Let's add to the action $\delta S = c \int d^4x\, F_{\mu\nu} \tilde{F}^{\mu\nu}$, which we can rewritte as:

$$\int d^4x\, \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \;\propto\; \int d^4x\, \epsilon^{\mu\nu\rho\sigma} \partial_\mu A_\nu \partial_\rho A_\sigma$$

$$\propto \int d^4x\, \partial_\mu \left(\epsilon^{\mu\nu\rho\sigma} A_\nu \partial_\rho A_\sigma \right) = \text{total derivative}$$

$\rightarrow$ it doesn't contribute to the equations of motion

2. Chern - Simons theory

$$S' = \int d^3x \left[\underbrace{-\frac{1}{4} G_{\mu\nu} G^{\mu\nu}}_{S_1'} + \underbrace{g \varepsilon^{\mu\nu\rho} B_\mu \partial_\nu B_\rho}_{S_2'} \right]$$

① As in the previous exercise, we do dimensional analysis:

$$[d^3x] = [M]^{-3}$$

$$[G] = [\partial][B] = [M]^1 \cdot [M]^\alpha$$

$$\rightsquigarrow [G^2] = [M]^2 [M]^{2\alpha} \rightsquigarrow 2\alpha + 2 - 3 = 0$$

$$\rightsquigarrow \alpha = \frac{1}{2} \rightarrow [B] = [M]^{1/2}$$

$$\rightsquigarrow \frac{1}{2} + \frac{1}{2} + 1 - 3 + \beta = 0 \quad \left([g] = [M]^\beta\right)$$

$$\rightsquigarrow \beta = 1.$$

$$\rightsquigarrow [g] = [M]$$

② Under $B_\mu \to B'_\mu = B_\mu + \partial_\mu \alpha$, we observe that

⊗ $S_1'' = S_1'$

$$\otimes,\ S_2' = g\int d^3x\, \varepsilon^{\mu\nu\lambda} B_\mu' \partial_\nu B_\lambda'$$

$$= g\int d^3x\, \varepsilon^{\mu\nu\lambda} (B_\mu + \partial_\mu \alpha)\, \partial_\nu\, (B_\lambda + \partial_\lambda \alpha)$$

$$= g\int d^3x\, \varepsilon^{\mu\nu\lambda} \Big(B_\mu \partial_\nu B_\lambda + B_\mu \cancelto{0}{\partial_\nu \partial_\lambda \alpha} + \partial_\mu \alpha\, \partial_\nu A_\lambda$$

$$+ \partial_\mu \alpha\, \cancelto{0}{\partial_\nu \partial_\lambda \alpha}\Big)$$

$$= g\int d^3x \left[\varepsilon^{\mu\nu\lambda} B_\mu \partial_\nu B_\lambda + \varepsilon^{\mu\nu\lambda} \partial_\mu \alpha\, \partial_\nu B_\lambda\right]$$

Now, $\partial_\mu \left(\varepsilon^{\mu\nu\lambda} \alpha\, \partial_\nu B_\lambda\right) = \varepsilon^{\mu\nu\lambda} \partial_\mu \alpha\, \partial_\nu B_\lambda$

$$+ \varepsilon^{\mu\nu\lambda} \alpha\, \cancelto{0}{\partial_\mu \partial_\nu B_\lambda}$$

$$= \varepsilon^{\mu\nu\lambda} \partial_\mu \alpha\, \partial_\nu B_\lambda$$

$$\leadsto S_2' = g\int d^3x \left[\varepsilon^{\mu\nu\lambda} B_\mu \partial_\nu B_\lambda + \partial_\mu \left(\varepsilon^{\mu\nu\lambda} \alpha\, \partial_\nu B_\lambda\right)\right]$$

$$= S_2 + \underbrace{g \oint dS_\mu\, \varepsilon^{\mu\nu\lambda} \alpha\, \partial_\nu B_\lambda}_{=0 \text{ at the bounderies.}}$$

(3) **Eom:**

$$\delta_B S' = \int d^3x \left[- G^{\mu\nu} \partial_\mu \delta B_\nu + g \varepsilon^{\mu\nu\lambda} \left(\delta B_\mu \partial_\nu B_\lambda + B_\mu \partial_\nu \delta B_\lambda \right) \right]$$

$$= \int d^3x \left[\partial_\mu G^{\mu\nu} \delta B_\nu + g \varepsilon^{\mu\nu\lambda} \left(\partial_\nu B_\lambda \delta B_\mu - \partial_\nu B_\mu \delta B_\lambda \right) \right]$$

$$= \int d^3x \left[\partial_\mu G^{\mu\nu} \delta B_\nu + g \varepsilon^{\mu\nu\lambda} \partial_\nu B_\lambda \delta B_\mu - g \varepsilon^{\mu\nu\lambda} \partial_\nu B_\mu \delta B_\lambda \right]$$

$$= \int d^3x \left[\partial_\alpha G^{\alpha\beta} \delta B_\beta - g \varepsilon^{\beta\gamma\alpha} \partial_\alpha B_\gamma \delta B_\beta - g \varepsilon^{\gamma\alpha\beta} \partial_\alpha B_\gamma \delta B_\beta \right]$$

$$= \int d^3x \left[\partial_\alpha G^{\alpha\beta} - 2g \varepsilon^{\alpha\beta\gamma} \partial_\alpha B_\gamma \right] \delta B_\beta .$$

$$\rightarrow \boxed{\partial_\alpha G^{\alpha\beta} - 2g \varepsilon^{\alpha\beta\gamma} \partial_\alpha B_\gamma = 0}.$$

We can do better though:

$$\varepsilon^{\alpha\beta\gamma} G_{\alpha\gamma} = \varepsilon^{\alpha\beta\gamma} \left(\partial_\alpha B_\gamma - \partial_\gamma B_\alpha \right) = 2 \varepsilon^{\alpha\beta\gamma} \partial_\alpha B_\gamma$$

→ eom equivalently written as:

$$\boxed{\partial_\alpha G^{\alpha\beta} - g \varepsilon^{\alpha\beta\gamma} G_{\alpha\gamma} = 0} \quad (*)$$

Since $G'_{\mu\nu} = G_{\mu\nu}$, the eom are trivially gauge-invariant.

4) consider now $\tilde{G}_\mu = \frac{1}{2}\varepsilon_{\mu\nu\lambda} G^{\nu\lambda}$

$$\varepsilon^{\mu\alpha\beta}\tilde{G}_\mu = \frac{1}{2}\varepsilon^{\mu\alpha\beta}\varepsilon_{\mu\nu\lambda} G^{\nu\lambda} = \frac{1}{2}\left(\delta_{\alpha\nu}\delta_{\beta\lambda} - \delta_{\alpha\lambda}\delta_{\beta\nu}\right) G^{\nu\lambda}$$

$$= \frac{1}{2}\left(G_{\alpha\beta} - G_{\beta\alpha}\right) = G_{\alpha\beta}.$$

Plugging this result into $(*)$, we find:

$$\rightsquigarrow \varepsilon^{\mu\alpha\beta}\partial_\alpha \tilde{G}_\mu + \tilde{g}\tilde{G}_\beta = 0, \quad (**) \quad \text{with } \tilde{g} = 2g$$

Let's now act on the above with $\varepsilon_{\nu\lambda\beta}$

$$\rightsquigarrow \varepsilon_{\nu\lambda\beta}\left(\varepsilon^{\mu\alpha\beta}\partial_\alpha \tilde{G}_\mu + \tilde{g}\tilde{G}_\beta\right) = 0$$

$$\rightsquigarrow \partial_\lambda \tilde{G}_\nu - \partial_\nu \tilde{G}_\lambda + \tilde{g}\varepsilon_{\nu\lambda\varkappa}\tilde{G}^\varkappa = 0$$

Hitting this expression with ∂^λ, we find:

$$\partial^2 \tilde{G}_\nu - \cancelto{0}{\partial_\nu \partial_\lambda \tilde{G}^\lambda} + \tilde{g}\varepsilon_{\nu\lambda\varkappa}\partial^\lambda \tilde{G}^\varkappa = 0,$$

where the second term vanishes by virtue of the Bianchi identity $\partial_\lambda \tilde{G}^\lambda = 0$ (it follows from $(**)$)

$$\rightsquigarrow \partial^2 \tilde{G}_\nu + \tilde{g}\varepsilon_{\nu\lambda\varkappa}\partial^\lambda \tilde{G}^\varkappa = 0$$

Using (**) to replace the second term in the above, we finally find: 7/7

$$\boxed{\partial^2 \tilde{G}_\nu + \tilde{g}^2 \tilde{G}_\nu = 0}$$

$\hookrightarrow$ eom for massive vector field, with $m_G^2 = \tilde{g}^2 = 4g^2$.

(5) The counting of degrees of freedom goes as follows:

- Initially we have 3 dof in B_μ.
- Due to the $U(1)$ invariance $B_\mu \to B'_\mu = B_\mu + \partial_\mu \alpha$ 1 is killed.
- On top of that, we also have got another constraint, the Bianchi identity $\partial_\mu \tilde{G}^\mu = 0$. Thus, another dof is killed.