Quantum Field Theory (Quantum Electrodynamics)

Problem Set 3 6 & 8 November 2023

1. Liouville theory

The action capturing the dynamics of the Liouville model in two spacetime dimensions reads

$$S = \int \mathrm{d}^2x \left[\frac{1}{2}(\partial_\mu \phi)^2 - a e^{b\phi}\right] ,$$

with $\mu = 0, 1$, ϕ a real scalar field, and a, b constants.

1. Find the equations of motion.
2. Find the analog of scale symmetry for this model.
3. Construct the conserved Noether current.

2. Maxwell theory

Consider the action of a vector field coupled to a source in a four-dimensional spacetime

$$S = \int \mathrm{d}^4x \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + A_\mu j^\mu\right) ,$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ the Maxwell field-strength tensor.

1. What are the dimensions of A_μ and j_μ ?
2. Derive the equations of motion.
3. What is the constraint on j_μ such that the action is invariant under the gauge transformation $A_\mu \to A'_\mu = A_\mu + \partial_\mu \alpha$?
4. Consider the dual field strength tensor $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\sigma\rho}F_{\sigma\rho}$, with $\epsilon^{\mu\nu\sigma\rho}$ the four-dimensional totally antisymmetric symbol ($\epsilon^{0123} = 1$). Is this quantity gauge invariant ?

   Add to the action the following term

   $$\delta S = c \int \mathrm{d}^4x F_{\mu\nu}\tilde{F}^{\mu\nu} ,$$

   with c a constant. Derive the equations of motion for $S + \delta S$. How are they modified due to presence of the new term ?

3. Chern-Simons theory

Consider a real vector field B_μ in a three-dimensional spacetime with action

$$S = \int \mathrm{d}^3x \left(-\frac{1}{4} G_{\mu\nu} G^{\mu\nu} + g \epsilon^{\mu\nu\rho} B_\mu \partial_\nu B_\rho \right) ,$$

where $\epsilon^{\mu\nu\rho}$ is the three-dimensional totally antisymmetric symbol ($\epsilon^{012} = 1$), g a real constant, and $G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$ its field-strength tensor.

1. Find the dimensions of B_μ and g.
2. Is the action invariant under the gauge transformation $B_\mu \to B'_\mu = B_\mu + \partial_\mu \alpha$?
3. Compute the equations of motion. Are they gauge invariant ?
4. Express the equations of motion in terms of the dual field $\tilde{G}^\mu = \frac{1}{2} \epsilon^{\mu\nu\rho} G_{\nu\rho}$. Do you recognize their form ? Explain.
5. What is the number of the propagating degrees of freedom in this theory ?
   Hint : The dual field satisfies
   $$\partial_\mu \tilde{G}^\mu = 0 \ .$$