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1. Klein–Gordon equation

(a) 1. $$0 = \delta_\phi S = \int d^4x \left[\partial_\mu \phi \partial^\mu \delta\phi - m^2 \phi \delta\phi \right]$$

$$= -\int d^4x \left[\partial^2 \phi + m^2 \phi \right] \delta\phi .$$

Since the variation should vanish $\forall \delta\phi$, we end up with the Klein–Gordon equation

$$(\partial^2 + m^2)\phi = 0 . \qquad \circledast$$

2. $$(\partial^2 + m^2) e^{i(\vec{p}\vec{x} \pm \omega_{\vec{p}} t)} = (\partial_0^2 - \partial_i^2 + m^2) e^{i(\vec{p}\vec{x} \pm \omega_{\vec{p}} t)}$$

$$= (-\omega_{\vec{p}}^2 + \vec{p}^2 + m^2) e^{i(\vec{p}\vec{x} \pm \omega_{\vec{p}} t)}$$

Therefore, the plane wave is a solution to $\circledast$, if the dispersion relation is

$$\omega_{\vec{p}} = \sqrt{\vec{p}^2 + m^2} .$$

The most general solution reads:

$$\phi(\vec{x}, t) = \int d^3\vec{p} \; e^{i\vec{p}\cdot\vec{x}} \tilde{\phi}(\vec{p}, t)$$

Plugging the above into $\circledast$, we find

$$(\partial_0^2 + \vec{p}^{\,2} + m^2)\tilde{\phi}(\vec{p},t) = 0 ,$$

with general solution

$$\tilde{\phi}(\vec{p},t) = \frac{g(p)}{(2\pi)^3}\left(a(\vec{p})\, e^{i\vec{p}\cdot\vec{x} - i\omega_{\vec{p}} t} + b(\vec{p})\, e^{i\vec{p}\cdot\vec{x} + i\omega_{\vec{p}} t}\right)$$

3. We start from

$$d^4p\ \delta(p^2 - m^2)\,\theta(p^0) = d^4p\ \delta((p^0)^2 - \vec{p}^{\,2} - m^2)\,\theta(p_0)$$

using the identity from the hint, the above is rewritten as:

$$d^4p \left(\frac{\delta(p^0 - \omega_{\vec{p}})}{2\omega_{\vec{p}}} + \frac{\delta(p^0 + \omega_{\vec{p}})}{2\omega_{\vec{p}}} \right)\theta(p^0)$$

Integrating the above over all momenta, we see

$$\int d^3\vec{p}\, dp^0 \left(\frac{\delta(p^0 - \omega_{\vec{p}})}{2\omega_{\vec{p}}} + \frac{\delta(p^0 + \omega_{\vec{p}})}{2\omega_{\vec{p}}} \right)\theta(p^0)$$

$$= \int \frac{d^3\vec{p}}{2\omega_{\vec{p}}}$$

The above implies that

$$\phi(\vec{x},t) = \int \frac{d^3\vec{p}}{2(2\pi)^3\,\omega_{\vec{p}}}\left(a(\vec{p})\, e^{i\vec{p}\vec{x} - i\omega_{\vec{p}} t} + b(\vec{p})\, e^{i\vec{p}\vec{x} + i\omega_{\vec{p}} t}\right) \quad \text{(**)}$$

4. Imposing that the field is real means

$$\phi^*(\vec{x},t) = \phi(\vec{x},t) \quad , \quad \text{with}$$

$$\phi^*(\vec{x},t) = \int \frac{d^3\vec{p}}{2(2\pi)^3 w_{\vec{p}}} \left(a^*(\vec{p})\, e^{-i\vec{p}\vec{x}+iw_{\vec{p}}t} + b^*(\vec{p})\, e^{-i\vec{p}\vec{x}-iw_{\vec{p}}t} \right)$$

$$= \int \frac{d^3\vec{p}}{2(2\pi)^3 w_{\vec{p}}} \left(a^*(\vec{p})\, e^{-i\vec{p}\vec{x}+iw_{\vec{p}}t} + b^*(-\vec{p})\, e^{i\vec{p}\vec{x}-iw_{\vec{p}}t} \right),$$

where in passing to the second line, the integration variable in the second term was changed $\vec{p} \to -\vec{p}$ (it's a dummy variable).

Comparing the above with (**), we immediately find

$$\boxed{b^*(-\vec{p}) = a(\vec{p})}$$

Thus:

$$\phi(\vec{x},t) = \int \frac{d^3\vec{p}}{2(2\pi)^3 w_{\vec{p}}} \left(a(\vec{p})\, e^{i\vec{p}\vec{x}-iw_{\vec{p}}t} + a^*(\vec{p})\, e^{i\vec{p}\vec{x}+iw_{\vec{p}}t} \right)$$

5. We are asked to compute the Hamiltonian density, which is the Legendre transform of the Lagrangian, i.e.

$$\mathcal{H} = \pi\dot{\phi} - \mathcal{L}$$

with $\pi \equiv \frac{\delta \mathcal{L}}{\delta \dot{\phi}} = \dot{\phi}$ the canonical momentum.

Thus:

$$\rightarrow \boxed{\mathcal{H} = \frac{\pi^2}{2} + \frac{1}{2}(\partial_i \phi)^2 + \frac{m^2}{2}\phi^2}$$

(b) 1. In the presence of the source ρ, the Klein-Gordon equation is easily found to be

$$(\partial^2 + m^2)\phi = \rho.$$

2. For a static point source, the above becomes

$$(\partial_i^2 - m^2)\phi(\vec{x}) = q\,\delta(\vec{x}). \quad (***)$$

In Fourier space (see point (a) as well)

$$\phi(\vec{x}) = \int d^3\vec{p}\, e^{i\vec{p}\vec{x}}\,\tilde{\phi}(\vec{p}),$$

$$\delta(\vec{x}) = \int d^3\vec{p}\, e^{i\vec{p}\vec{x}}$$

From $\cancel{(**)}$, we find

$$\tilde{\phi}(\vec{p}) = \frac{-q}{\vec{p}^2+m^2}$$, so moving to coordinate space

$$\Rightarrow \phi(\vec{x}) = \int d^3\vec{p}\, e^{i\vec{p}\cdot\vec{x}}\tilde{\phi}(\vec{p}) = -q\int d^3\vec{p}\,\frac{e^{i\vec{p}\cdot\vec{x}}}{\vec{p}^2+m^2}$$

$$= -q\int dp\, p^2\, d\theta \sin\theta\, d\phi\, \frac{e^{ipx\cos\theta}}{p^2+m^2}$$

Up to irrelevant numerical factors, we find

$$\Rightarrow \phi(\vec{x}) \propto -\frac{q}{x}\int \frac{dp\, p\sin(px)}{p^2+m^2}$$

$$= -\frac{q}{2ix}\int_0^\infty \frac{dp\, p}{p^2+m^2}\left(e^{ipx}-e^{-ipx}\right)$$

$$= -\frac{q}{2ix}\left[\int_0^\infty \frac{dp\, p}{p^2+m^2}e^{ipx} + \int_{-\infty}^0 \frac{dp\, p\, e^{ipx}}{p^2+m^2}\right]$$

$$= \frac{-q}{2ix}\int_{-\infty}^{+\infty}\frac{dp\, p\, e^{ipx}}{(p+im)(p-im)}$$

$$\Rightarrow \phi(\vec{x}) = \frac{-q}{2ix}(2\pi i)\sum \text{residues}$$

Define $$f(z) = \frac{z e^{izx}}{(z+im)(z-im)}$$

$\circledast$ $\mathrm{Res}[f(z), z=-im] = \lim_{z\to +im} (z-im) f(z)$

$$= \lim_{z\to +im} \frac{ze^{izx}}{z+im} = \frac{e^{-mx}}{2}.$$

$\circledast$ $\mathrm{Res}[f(z), z=+im] = \lim_{z\to -im} (z+im) f(z)$

$$= \lim_{z\to -im} \frac{z\, e^{izx}}{z-im} = \frac{e^{-mx}}{2}$$

Thus, $\varphi(\vec{x}) = -\frac{q}{x} e^{-mx} \Rightarrow$ Yukawa potential.

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2. Scale symmetry (dilatations)

1. Equation of motion:

$$\partial^2\phi + \lambda\phi^3 = 0 \quad \circledast$$

2. $\phi(x) \to \varphi'(x) = \alpha\phi(\alpha x)$

$$\partial_\mu \varphi(x) = \frac{\partial}{\partial x^\mu}\varphi(x) \to \frac{\partial}{\partial x^\mu}(\alpha\varphi(\alpha x))$$

$$= \alpha\frac{\partial}{\partial x^\mu}\varphi(\alpha x) = \alpha^2\frac{\partial}{\partial(\alpha x^\mu)}\phi(\alpha x)$$

Introducing $y^\mu = \alpha x^\mu$, we see that

$$\frac{\partial}{\partial x^\mu}\phi(x) = \alpha^2 \frac{\partial}{\partial y^\mu}\phi(y)$$

$$\rightsquigarrow S' = \int d^4x \left(\frac{1}{2}\alpha^4\, \partial_\mu\varphi(y)\partial_\mu\phi(y) - \frac{\lambda}{4}\alpha^4\varphi^4(y)\right)$$

$$= \int d^4(\alpha x)\left(\frac{1}{2}\partial_\mu\varphi(y)\,\partial_\mu\varphi(y) - \frac{\lambda}{4}\phi^4(y)\right)$$

$$= \int d^4y\left(\frac{1}{2}\partial_\mu\varphi\,\partial_\mu\varphi - \frac{\lambda}{4}\varphi^4\right) = S$$

3. Now we look for the current.

Consider the infinitesimal transformation,

$$\alpha = 1 - a$$

$$\rightsquigarrow \varphi'(x) = (1-a)\,\varphi(x - ax)$$

$$= \varphi(x - ax) - a\,\varphi(x - ax)$$

$$= \phi(x) - a x^\mu \partial_\mu\phi - a\,\phi(x) + \mathcal{O}(a^2)$$

$$\rightsquigarrow \delta\varphi = -a\left(\varphi(x) + x^\mu\partial_\mu\phi\right)$$

From Noether's theorem, the above gives rise to the following current:

$$j_\mu = \frac{\delta \mathcal{L}}{\delta \partial_\mu \varphi} \delta \varphi + \varepsilon_\mu \mathcal{L}$$

$$= -\partial_\mu \phi \left(\phi + x^\nu \partial_\nu \phi\right) + \varepsilon_\mu \mathcal{L}$$

$$= -\phi \partial_\mu \varphi - x^\nu \partial_\mu \varphi \partial_\nu \varphi + x_\mu \mathcal{L}$$

$$\boxed{\rightarrow j_\mu = -\varphi \partial_\mu \varphi - x^\nu \partial_\nu \phi \partial_\mu \varphi + x_\mu \left(\frac{(\partial_\nu \varphi)^2}{2} - \frac{\lambda}{4} \varphi^4\right)}$$

Let's check if it is indeed conserved:

$$\partial_\mu j^\mu = \cancel{-(\partial\phi)^2} - \phi \partial^2 \phi \cancel{-(\partial\phi)^2} - x^\nu \partial_\mu \partial_\nu \phi \partial^\mu \phi$$

$$- x^\nu \partial_\nu \phi \partial^2 \phi + 2\cancel{(\partial\phi)^2} - \lambda \phi^4$$

$$+ x_\mu \partial^\mu \left(\frac{(\partial_\nu \varphi)^2}{2} - \frac{\lambda}{4} \phi^4\right)$$

$$= \left(-\phi + x^\nu \partial_\nu \phi\right)\left(\partial^2 \phi + \lambda \phi^3\right) = 0,$$

on the equation of motion $\circledast$.

To derive the form of the current, we use the following trick. Take $a = a(x)$ instead of constant & find the response of the action w.r.t. the transformation.

$$\delta S' = \int d^4x \left\{ \partial_\mu \varphi \, \partial^\mu \delta\varphi - \lambda \varphi^3 \delta\varphi \right\}$$

$$= -\int d^4x \left\{ \partial_\mu \varphi \, \partial^\mu [a(\varphi + x^\nu \partial_\nu \varphi)] - \lambda \varphi^3 a (\varphi + x^\nu \partial_\nu \varphi) \right\}$$

$$= -\int d^4x \Big\{ \partial_\mu \varphi \left[(\varphi + x^\nu \partial_\nu \varphi) \partial^\mu a + a \, \partial^\mu (\varphi + x^\nu \partial_\nu \varphi) \right]$$
$$- a \lambda \varphi^4 - a \lambda \varphi^3 x^\nu \partial_\nu \varphi \Big\}$$

$$= -\int d^4x \Big\{ (\varphi + x^\nu \partial_\nu \varphi) \partial_\mu \varphi \, \partial^\mu a + a (\partial_\mu \varphi)^2 + a \, \partial_\mu \varphi \, \partial^\mu x^\nu \partial_\nu \varphi$$
$$+ a \, \partial_\mu \varphi \, x^\nu \partial^\mu \partial_\nu \varphi - a \lambda \varphi^4 - \frac{\lambda}{4} a \, x^\nu \partial_\nu \varphi^4 \Big\}$$

$$= -\int d^4x \Big\{ (\varphi + x^\nu \partial_\nu \varphi) \partial_\mu \varphi \, \partial^\mu a + 2a (\partial_\mu \varphi)^2$$
$$+ \underline{\frac{a}{2} x^\nu \partial_\nu (\partial_\mu \varphi)^2} - a \lambda \varphi^4 - \underline{\frac{\lambda}{4} a \, x^\nu \partial_\nu \varphi^4} \Big\}.$$

We now integrate by parts the underlined terms, to obtain (after dropping full derivatives)

$$\delta S' = -\int d^4x \Big\{ (\varphi + x^\nu \partial_\nu \varphi) \partial_\mu \varphi \, \partial^\mu a + 2a (\partial_\mu \varphi)^2$$
$$- \frac{1}{2} \partial_\nu (a x^\nu) (\partial_\mu \varphi)^2 - a \lambda \varphi^4 + \frac{\lambda}{4} \varphi^4 \partial_\nu (a x^\nu) \Big\}$$

$$= -\int d^4x \Big\{ (\varphi + x^\nu \partial_\nu \varphi) \partial_\mu \varphi \, \partial^\mu a - \frac{1}{2} (\partial_\nu \varphi)^2 x_\mu \partial^\mu a$$
$$+ \frac{\lambda}{4} \varphi^4 x_\mu \partial^\mu a \Big\}$$

$$\rightarrow \delta S' = \int d^4x\, \partial^\mu a \left[-(\varphi + x^\nu \partial_\nu \varphi)\partial_\mu \varphi + x_\mu \mathcal{L} \right],$$

with

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)^2 - \frac{\lambda}{4}\varphi^4 \quad .$$

Therefore, the Noether current is

$$j_\mu = -(\varphi + x^\nu \partial_\nu \varphi)\partial_\mu \varphi + x_\mu \mathcal{L},$$

as expected.