Quantum Field Theory (Quantum Electrodynamics)

Problem Set 2 30 & 31 October 2023

1. Klein-Gordon equation

a) Consider the action of a real massive scalar field in four spacetime dimensions

$$S = \int \mathrm{d}^4x \left[\frac{1}{2}(\partial_\mu\phi)^2 - \frac{m^2}{2}\phi^2\right] \ .$$

1. Find the equation of motion (Klein-Gordon equation) for the scalar field.
2. Show that a particular solution of the Klein-Gordon equation is a plane wave of the form
$$e^{i\vec{p}\cdot\vec{x}\pm i\omega_{\vec{p}}t} \ ,$$
with $\omega_{\vec{p}} = +\sqrt{\vec{p}^2+m^2}$. Argue that the general solution reads
$$\phi(\vec{x},t) = \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3 f(p)} \left[a(\vec{p})e^{i\vec{p}\cdot\vec{x}-i\omega_{\vec{p}}t} + b(\vec{p})e^{i\vec{p}\cdot\vec{x}+i\omega_{\vec{p}}t}\right] \ ,$$
where $a(\vec{p})$ and $b(\vec{p})$ are arbitrary functions of the wave vector $\vec{p}$, and $f(p)$ is a function of the magnitude of p.
3. Show that the measure of integration appearing in the above is Lorentz-invariant provided that $f(p) = 2\omega_{\vec{p}}$.
Hint : Start from the Lorentz-invariant expression
$$\mathrm{d}^4p\, \delta(p_\mu p^\mu - m^2)\theta(p^0) \ ,$$
and use the identity
$$\delta(x^2 - x_0^2) = \frac{1}{2|x|}\Big(\delta(x-x_0) + \delta(x+x_0)\Big) \ .$$
4. Find the relation between $a(\vec{p})$ and $b(\vec{p})$ which follows by demanding that $\phi(\vec{x},t)$ is a real field.
5. Compute the Hamiltonian density in terms of the field $\phi(x)$ and its conjugate momentum $\pi(x)$.

b) Consider again the action of a real massive scalar field in four spacetime dimensions but now with a source term $\rho(x)$

$$S = \int \mathrm{d}^4x \left[\frac{1}{2}(\partial_\mu\phi)^2 - \frac{m^2}{2}\phi^2 + \rho\phi\right] \ .$$

1. Show how the equations of motion for the scalar field are modified as compared to the previous case.
2. Solve them for a static point source $\rho(\vec{x}) = -q\delta^{(3)}(\vec{x})$.

2. Scale symmetry (dilatations)

Consider the action of a real massless scalar field in four spacetime dimensions

$$S = \int \mathrm{d}^4x \left[\frac{1}{2}(\partial_\mu\phi)^2 - \frac{\lambda}{4}\phi^4\right] ,$$

with λ a real constant.

1. Find the equation of motion.
2. Show that the action is invariant under dilatations

$$\phi(x) \to \phi'(x) = \alpha\phi(\alpha x) ,$$

with α a real constant parameter.

3. Find the corresponding Noether current. Show that it is conserved on the equations of motion.
Hint : Consider the infinitesimal dilatation

$$\alpha \approx 1 - a , \quad a \ll 1 ,$$

and show that transformation of the field is

$$\delta\phi \equiv \phi'(x) - \phi(x) = -a(\phi(x) + x^\mu\partial_\mu\phi(x)) .$$

3. Liouville theory

The action capturing the dynamics of the Liouville model in two spacetime dimensions reads

$$S = \int \mathrm{d}^2x \left[\frac{1}{2}(\partial_\mu\phi)^2 - ae^{b\phi}\right] ,$$

with $\mu = 0, 1$, ϕ a real scalar field, and a, b constants.

1. Find the equations of motion.
2. Find the analog of scale symmetry for this model.
3. Construct the conserved Noether current.