QFT (QED) – Problem Set 1

1. Planck units

ⓐ

$[c] = [L T^{-1}]$, $[G] = [L^3 M^{-1} T^{-2}]$,

$[\hbar] = [L^2 M T^{-1}]$, $[k_B] = [M L^2 \Theta^{-1} T^{-2}]$.

⊛ Dimensions of length

$$\left[\frac{\hbar G}{c^3}\right] = [L^2] \Rightarrow l_{Pl} = \sqrt{\frac{\hbar G}{c^3}} \approx 10^{-33}\,\text{cm} \approx 10^{-20}\,\text{GeV}^{-1}$$

⊛ Dimensions of time

$$\left[\frac{\hbar G}{c^5}\right] = [T^2] \Rightarrow t_{Pl} = \sqrt{\frac{\hbar G}{c^5}} \approx 10^{-44}\,\text{s} \approx 10^{-20}\,\text{GeV}^{-1}$$

⊛ Dimensions of mass

$$\left[\frac{\hbar c}{G}\right] = [M^2] \Rightarrow m_{Pl} = \sqrt{\frac{\hbar c}{G}} \approx 10^{-5}\,\text{g} \approx 10^{-19}\,\text{GeV}$$

⊛ Dimensions of temperature

$$\left[\frac{\hbar c^5}{G k_B^2}\right] = [\Theta^2] \Rightarrow T_{Pl} = \sqrt{\frac{\hbar c^5}{G k_B^2}} \approx 10^{32}\,\text{K}.$$

ⓑ $$[a] = \left[\frac{L}{T^2}\right] \Rightarrow a_{Pl} = \frac{l_{Pl}}{t_{Pl}^2} \approx \frac{10^{-35}\,\text{m}}{10^{-88}\,\text{s}^2} \times \frac{g}{g}$$

$$\Rightarrow a_{Pl} \approx 10^{50}\,g$$

$$E_{ll} = \frac{m_{ll}\, l_{ll}^2}{t_{ll}^2} \approx 10^{19}\ GeV$$

ⓒ $\lambda_{el} = \frac{1}{m_{el}}$, $m_{el} \approx 0{,}5\ MeV$

$\rightarrow \lambda_{el} \approx MeV^{-1} \approx 10^{-6}\ eV^{-1}$

& $\lambda_{el} \approx 10^{-9}\ cm$

2. Fun with indices

① $A_\mu A^\mu = \eta_{\mu\nu} A^\mu A^\nu = \eta_{00}(A^0)^2 + \eta_{11}(A^1)^2 + \eta_{22}(A^2)^2 + \eta_{33}(A^3)^2$

$= (A^0)^2 - (A^1)^2 - (A^2)^2 - (A^3)^2$, since

$\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$

② consistent : $A^\alpha{}_\beta = B^\alpha{}_\gamma C^\gamma{}_\delta D^\delta{}_\beta$

inconsistent : $A^\alpha{}_\beta = B^\alpha{}_\gamma C^\gamma{}_\delta D^\delta{}_\rho$

$A^\alpha{}_\beta = B^\alpha{}_\gamma C^\gamma{}_\gamma D^\gamma{}_\beta$

③ $\delta_\mu{}^\mu = Tr(\mathbb{1}_{n\times n}) = n$

4) $\xi_{\mu\nu} A^{\mu\nu} = -\xi_{\mu\nu} A^{\nu\mu}$

Since all indices are summed, we can rename them and end up with:

$$\xi_{\rho\sigma} A^{\rho\sigma} = -\xi_{\rho\sigma} A^{\rho\sigma} \Rightarrow \xi_{\rho\sigma} A^{\rho\sigma} = 0$$

5) $B_{\mu\nu} = \frac{1}{2}(B_{\mu\nu} + B_{\nu\mu}) + \frac{1}{2}(B_{\mu\nu} - B_{\nu\mu}) = B_{(\mu\nu)} + B_{[\mu\nu]}$,

where $(\ldots)$ & $[\ldots]$ denote symmetrization & antisymmetrization of the corresponding indices.

$\Rightarrow \xi_{\mu\nu} B^{\mu\nu} = \xi_{\mu\nu} B^{(\mu\nu)} = \frac{1}{2} \xi_{\mu\nu} (B^{\mu\nu} + B^{\nu\mu})$, since

$\xi_{\mu\nu} B^{[\mu\nu]} = 0$.

6) $A_{\mu\nu} B^{\mu\nu} = A_{\mu\nu} B^{[\mu\nu]} = \frac{1}{2} A_{\mu\nu} (B^{\mu\nu} - B^{\nu\mu})$, since

$A_{\mu\nu} B^{(\mu\nu)} = 0$.

3. The totally antisymmetric symbol

1). $\varepsilon_{0123} = -1$

$$\epsilon^{\alpha\beta\gamma\delta} = \eta^{\alpha\alpha'}\eta^{\beta\beta'}\eta^{\gamma\gamma'}\eta^{\delta\delta'}\epsilon_{\alpha'\beta'\gamma'\delta'}$$

$$\rightarrow \epsilon^{0123} = \eta^{0\alpha'}\eta^{1\beta'}\eta^{2\gamma'}\eta^{3\delta'}\epsilon_{\alpha'\beta'\gamma'\delta'}$$

$$= \eta^{00}\eta^{11}\eta^{22}\eta^{33}\epsilon_{0123} = 1\times(-1)\times(-1)\times(-1)\times(-1)$$

$$= 1$$

(2) We start from

$$\epsilon_{\alpha_1\alpha_2\cdots\alpha_m\mu_1\mu_2\cdots\mu_n}\epsilon^{\alpha_1\alpha_2\cdots\alpha_m\nu_1\nu_2\cdots\nu_n} = (-1)^s m!\, n!\, \delta^{[\nu_1}_{[\mu_1}\cdots\delta^{\nu_n]}_{\mu_n]}$$

with $m=n=2$, $s=3$

$$\rightarrow \epsilon_{\alpha\beta\gamma\delta}\epsilon^{\varkappa\lambda\gamma\delta} = -2!\,2!\,\delta^{[\varkappa}_{[\alpha}\delta^{\lambda]}_{\beta]}$$

$$= -2!\,2!\,\frac{1}{2}\left(\delta^{\varkappa}_{[\alpha}\delta^{\lambda}_{\beta]} - \delta^{\lambda}_{[\alpha}\delta^{\varkappa}_{\beta]}\right)$$

$$= -2!\,2!\,\frac{1}{4}\,2\left(\delta^{\varkappa}_{\alpha}\delta^{\lambda}_{\beta} - \delta^{\lambda}_{\alpha}\delta^{\varkappa}_{\beta}\right)$$

$$= -2\left(\delta^{\varkappa}_{\alpha}\delta^{\lambda}_{\beta} - \delta^{\lambda}_{\alpha}\delta^{\varkappa}_{\beta}\right)$$

(3) using the above result, we see that

$$\epsilon^{\varkappa\lambda\mu\nu}\epsilon_{\varkappa\lambda\mu\nu} = -2\left(\delta^{\varkappa}_{\varkappa}\delta^{\lambda}_{\lambda} - \delta^{\lambda}_{\varkappa}\delta^{\varkappa}_{\lambda}\right) = -4!$$

reminder: $\partial_\mu \varphi$ is shorthand for $\frac{\partial \varphi}{\partial x^\mu}$

(4) Equations of motion

(1) $S_\varphi = \int d^4x \left[\frac{1}{2}(\partial_\mu \varphi)(\partial^\mu \varphi) - \frac{m^2}{2}\varphi^2 - \frac{\lambda}{4}\varphi^4 \right]$

We will use the Euler-Lagrange eq. for φ to derive the e.o.m. for φ:

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right) - \frac{\partial \mathcal{L}}{\partial \varphi} = 0$$

Let us do this first calulation in detail:

$$\frac{\partial \mathcal{L}}{\partial \varphi} = -m^2\varphi - \lambda\varphi^3$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} = \frac{\partial}{\partial(\partial_\mu \varphi)} \left(\frac{1}{2}(\partial_\alpha \varphi)(\partial_\beta \varphi)\eta^{\alpha\beta} \right) = \frac{1}{2}\left(\delta^\mu_\alpha (\partial_\beta \varphi)\eta^{\alpha\beta} + (\partial_\alpha \varphi)\delta^\mu_\beta \eta^{\alpha\beta} \right) =$$

$$= (\partial^\mu \varphi)$$

$$\Rightarrow \boxed{\partial_\mu \partial^\mu \varphi + m^2\varphi + \lambda\varphi^3 = 0.}$$

reminder: $\Box = \partial^\mu \partial_\mu = \partial_0^2 - \Delta$ is called the d'Alembert operator

(2) $S_A = \int d^4x \left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - A_\mu j^\mu \right]$

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

The Euler-Lagrange eq. for A_μ is $\partial_\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)} \right) - \frac{\partial \mathcal{L}}{\partial A_\mu} = 0$

$$\frac{\partial \mathcal{L}}{\partial A_\mu} = -j^\mu$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)} = -\frac{1}{4}\frac{\partial}{\partial(\partial_\nu A_\mu)} \left(F_{\alpha\beta}F_{\rho\sigma}\eta^{\rho\alpha}\eta^{\sigma\beta} \right)$$

$$\frac{\partial F_{\alpha\beta}}{\partial(\partial_\nu A_\mu)} = \delta^\nu_\alpha \delta^\mu_\beta - \delta^\nu_\beta \delta^\mu_\alpha$$

$$\Rightarrow \frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)} = -\frac{1}{4}\left(F^{\nu\mu} - \underbrace{F^{\mu\nu}}_{= \partial^\mu A^\nu - \partial^\nu A^\mu = -F^{\nu\mu}} \right) \cdot 2 = -F^{\nu\mu}$$

$$\Rightarrow \boxed{\partial_\nu F^{\nu\mu} = j^\mu.}$$

(3) $S_B = \int d^4x \left[-\frac{1}{4} G_{\mu\nu} G^{\mu\nu} + \frac{m^2}{2} B_\mu B^\mu \right]$

$G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$

Similarly to (2) we obtain:

$$\frac{\partial \mathcal{L}}{\partial B_\mu} = m^2 B^\mu$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\nu B_\mu)} = -G^{\nu\mu}$$

$$\Rightarrow \boxed{\partial_\nu G^{\nu\mu} + m^2 B^\mu = 0.}$$