Quantum Field Theory (Quantum Electrodynamics)

Problem Set 1 23 & 25 October 2023

1. Planck units

a) Derive the following elementary quantities : the Planck length $l_{\rm Pl}$ in cm, the Planck time $t_{\rm Pl}$ in s, the Planck mass $m_{\rm Pl}$ in g and the Planck temperature $T_{\rm Pl}$ in K.
Hint : Write the dimensions of c, $\hbar$, $G_{\rm N}$, $k_{\rm B}$ in terms of length L, time T, mass M and temperature Θ. Substitute with the basic Planck units $l_{\rm Pl}$, $t_{\rm Pl}$, $m_{\rm Pl}$, $T_{\rm Pl}$.

b) From those basic Planck units derive the Planck acceleration $a_{\rm Pl}$ in g (take $g = 9.81$ m/s^2) and Planck energy $E_{\rm Pl}$ in GeV.

c) In Planckian units we set $c = \hbar = G_{\rm N} = k_{\rm B} = 1$, so that any quantity expressed in these units is dimensionless. Calculate the Compton wavelength of the electron in Planckian units, in cm and in eV^{-1}.

2. Fun with indices

1. The scalar product of a vector field $A^\mu = (A^0, A^1, A^2, A^3)$ with itself is defined as

$$A_\mu A^\mu \equiv \eta_{\mu\nu} A^\mu A^\nu \ ,$$

   with $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ the 4 dimensional Minkowski metric. Write the above expression in terms of the vector components.

2. Give some examples of tensor equations consistent and inconsistent with the Einstein's summation convention.
3. Which is the value of δ^μ_μ in 4 dimensions ? And in n dimensions ?
4. Completely symmetric and antisymmetric rank-2 tensors satisfy $T_{\mu\nu} = \pm T_{\nu\mu}$, where the plus sign stands for the symmetric and the minus sign for the antisymmetric one. Show that, if $S_{\mu\nu}$ is a rank-2 symmetric tensor and $A_{\mu\nu}$ is a rank-2 antisymmetric tensor, then $S_{\mu\nu} A^{\mu\nu} = 0$.
5. Show that, if $S_{\mu\nu}$ is symmetric and $B_{\mu\nu}$ is arbitrary, $S_{\mu\nu} B^{\mu\nu} = \frac{1}{2} S_{\mu\nu} \left(B^{\mu\nu} + B^{\nu\mu} \right)$.
6. Show that, if $A_{\mu\nu}$ is antisymmetric and $B_{\mu\nu}$ is arbitrary, $A_{\mu\nu} B^{\mu\nu} = \frac{1}{2} A_{\mu\nu} \left(B^{\mu\nu} - B^{\nu\mu} \right)$.

3. The totally antisymmetric symbol

The generalization of the three-dimensional totally antisymmetric symbol in Minkowski spacetime is $\epsilon^{\kappa\lambda\mu\nu}$ and satisfies

$$\epsilon^{\kappa\lambda\mu\nu} = \begin{cases} +1 & \text{if } \kappa\lambda\mu\nu \text{ is an even permutation of } 0123 \ , \\ -1 & \text{if } \kappa\lambda\mu\nu \text{ is an odd permutation of } 0123 \ , \\ 0 & \text{otherwise} \ , \end{cases}$$

1. Show that if $\epsilon_{0123} = -1$, then $\epsilon^{0123} = 1$.
2. Show that $\epsilon_{\alpha\beta\gamma\delta}\epsilon^{\kappa\lambda\gamma\delta} = -2(\delta^\kappa_\alpha\delta^\lambda_\beta - \delta^\lambda_\alpha\delta^\kappa_\beta)$.
   Hint: Contractions of the totally antisymmetric tensor can be expressed in terms of products of delta functions as :
   $$\epsilon_{\alpha_1\alpha_2\ldots\alpha_m\mu_1\mu_2\ldots\mu_n}\epsilon^{\alpha_1\alpha_2\ldots\alpha_m\nu_1\nu_2\ldots\nu_n} = (-1)^s m!n!\delta^{[\nu_1}_{[\mu_1}\cdots\delta^{\nu_n]}_{\mu_n]} \ ,$$
   where s is the number of negative eigenvalues of the metric and $\delta^{[\nu_1\cdots\nu_n]}$ denotes a fully anti-symmetrized tensor $\delta^{\nu_1\cdots\nu_n}$ (and the same for lower indices).
3. Prove that $\epsilon_{\kappa\lambda\mu\nu}\epsilon^{\kappa\lambda\mu\nu} = -4!$.

4. Equations of motion

1. Consider the action of a self-interacting real massive scalar field ϕ
   $$S_\phi = \int d^4x \left[\frac{1}{2}(\partial_\mu\phi)^2 - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4}\phi^4\right] \ ,$$
   with m and λ constants. Derive the equations of motion for ϕ.
2. Consider the action of the electromagnetic field A_μ coupled to a source j_μ
   $$S_A = \int d^4x \left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - A_\mu j^\mu\right] \ ,$$
   with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, the Maxwell field-strength tensor. Derive the equations of motion for A_μ.
3. Consider the action of a massive vector field B_μ
   $$S_B = \int d^4x \left[-\frac{1}{4}G_{\mu\nu}G^{\mu\nu} + \frac{m^2}{2}B_\mu B^\mu\right] \ ,$$
   with $G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$. Derive the equations of motion for B_μ.