3. Feynman rules

- The Lagrangian reads

$$\tilde{\mathcal{L}} = -\frac{1}{4}F_{\mu\nu}^2 + i\bar{\psi}(\not\partial - m)\psi - e\bar{\psi}\not{A}\psi + i\bar{\Psi}(\not\partial - M)\Psi - e\bar{\Psi}\not{A}\Psi \,. \quad (1)$$

① The Feynman rules are:

photon propagator: μ ~~~~ ν (momentum k) $= \dfrac{-i\eta_{\mu\nu}}{k^2}$

fermion ψ —: $\xrightarrow{p}$ $= \dfrac{i(\not p + m)}{p^2 - m^2}$

fermion Ψ —: $\xRightarrow{p}$ $= \dfrac{i(\not p + M)}{p^2 - M^2}$

Interaction vertex of ψ with photon: $= -ie\gamma^\mu$

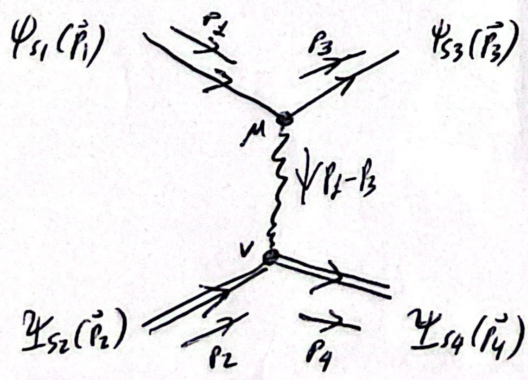
Interaction vertex of Ψ with photon: $= -ie\gamma^\mu$

4-momentum is conserved @ every vertex.

We also note that:

incoming ψ : $u_s^\psi(\vec{p})$	incoming Ψ : $u_s^\Psi(\vec{p})$
outgoing ψ : $\bar{u}_s^\psi(\vec{p})$	outgoing Ψ : $\bar{u}_s^\Psi(\vec{p})$
incoming $\bar{\psi}$: $\bar{v}_s(\vec{p})$	incoming $\bar{\Psi}$: $\bar{v}_s^\Psi(\vec{p})$
outgoing $\bar{\psi}$: $v_s(\vec{p})$	outgoing $\bar{\Psi}$: $v_s^\Psi(\vec{p})$

② Using the above, we write down the diagram corresponding to $\psi\Psi \to \psi\Psi$ scattering



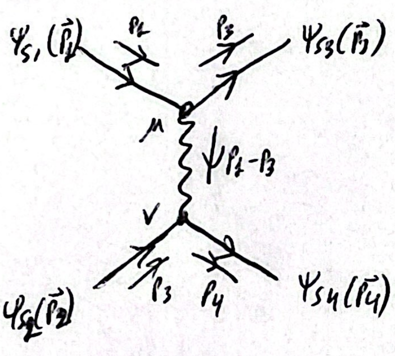
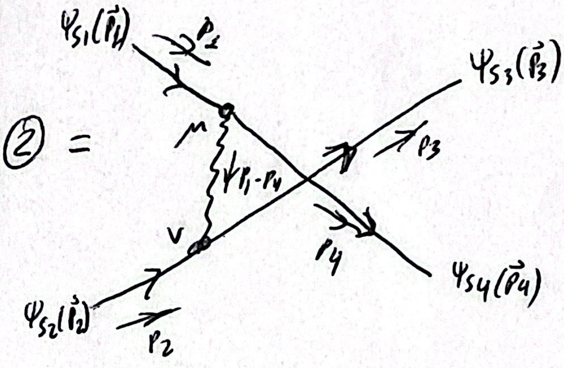
The corresponding amplitude reads

$$i\mathcal{M} = \bar{u}^{\psi}_{s_3}(\vec{p}_3)(-ie\gamma^{\mu})u^{\psi}_{s_1}(\vec{p}_1)\,\frac{(-i\eta_{\mu\nu})}{(p_1-p_3)^2}\,\bar{u}^{\psi}_{s_4}(\vec{p}_4)(-ie\gamma^{\nu})u^{\psi}_{s_2}(\vec{p}_2)$$

$$= \frac{ie^2}{t}\bar{u}^{\psi}_{s_3}(\vec{p}_3)\gamma^{\mu}u^{\psi}_{s_1}(\vec{p}_1)\bar{u}^{\psi}_{s_4}(\vec{p}_4)u^{\psi}_{s_2}(\vec{p}_2) \qquad (2)$$

as it should and in agreement with the previous exercise.

③ Now we take the special case $m=0$ & consider the $\psi\psi \to \psi\psi$ process. We have two diagrams,

① =  , ② = 

The amplitudes read

$$\circledast\ i\mathcal{M}_{①} = \bar{u}^{\psi}_{s_3}(\vec{p}_3)(-ie\gamma^{\mu})u^{\psi}_{s_1}(\vec{p}_1)\,\frac{(-i\eta_{\mu\nu})}{(p_1-p_3)^2}\,\bar{u}^{\psi}_{s_4}(\vec{p}_4)(-ie\gamma^{\nu})u^{\psi}_{s_2}(\vec{p}_2)$$

$$\to \mathcal{M}_{①} = \frac{e^2}{t}\bar{u}^{\psi}_{s_3}(\vec{p}_3)\gamma^{\mu}u^{\psi}_{s_1}(\vec{p}_1)\bar{u}^{\psi}_{s_4}(\vec{p}_4)\gamma_{\mu}u^{\psi}_{s_2}(\vec{p}_2) \quad . \qquad (3)$$

$$\circledast\ i\mathcal{M}_{②} = \bar{u}^{\psi}_{s_4}(\vec{p}_4)(-ie\gamma^{\mu})u^{\psi}_{s_1}(\vec{p}_1)\,\frac{(-i\eta_{\mu\nu})}{(p_1-p_4)^2}\,\bar{u}^{\psi}_{s_3}(\vec{p}_3)(-ie\gamma^{\nu})u^{\psi}_{s_2}(\vec{p}_2)$$

$$\to \mathcal{M}_{②} = \frac{e^2}{u}\bar{u}^{\psi}_{s_4}(\vec{p}_4)\gamma^{\mu}u^{\psi}_{s_1}(\vec{p}_1)\bar{u}^{\psi}_{s_3}(\vec{p}_3)\gamma_{\mu}u^{\psi}_{s_2}(\vec{p}_2) \quad . \qquad (4)$$

The total amplitude averaged & squared reads

$$|\overline{\mathcal{M}}|^2 = \frac{1}{4}\sum_{spins}|\mathcal{M}|^2 = \frac{1}{4}\sum_{spins}\left(|\mathcal{M}_{①}|^2 + |\mathcal{M}_{②}|^2 - \mathcal{M}_{①}\mathcal{M}_{②}^{\dagger} - \mathcal{M}_{①}^{\dagger}\mathcal{M}_{②}\right) , \qquad (5)$$

where we took into account that the two diagrams <u>interfere destructively</u>. This is because we are dealing with anticommuting particles and thus the total amplitude must be antisymmetric in the exchange of the final

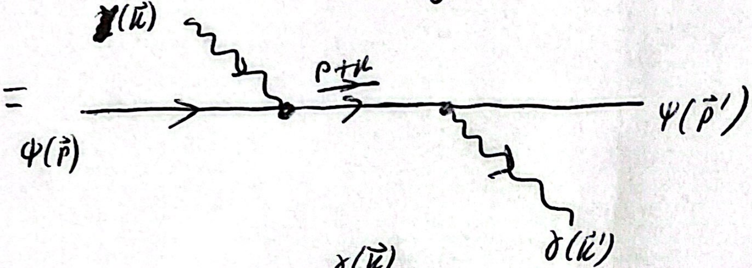
From eqs. (3,4) we get

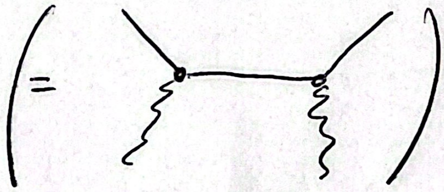
$$|\overline{\mathcal{M}}|^2 = 2e^4\left(\frac{s^2+u^2}{t^2} + \frac{s^2+t^2}{u^2} + \frac{2s^2}{tu}\right) \quad ,(6)$$

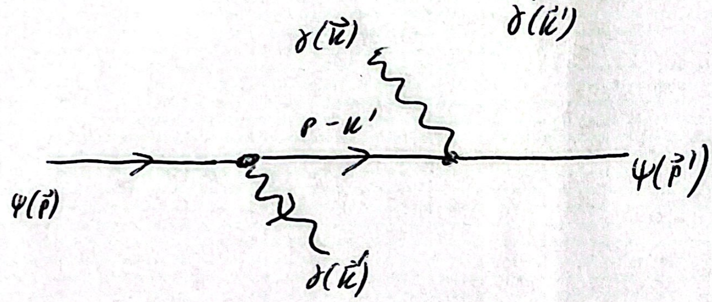
meaning that

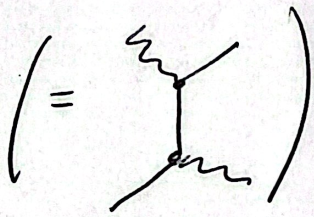
$$\frac{d\sigma}{d\Omega}(\psi\psi\to\psi\psi) = \frac{|\overline{\mathcal{M}}|^2}{64\pi s} = \frac{\alpha^2}{2s}\left(\frac{s^2+u^2}{t^2} + \frac{s^2+t^2}{u^2} + \frac{2s^2}{tu}\right) . (7)$$

(4) For Compton scattering, there are two diagrams also:

① = 

(= )

② = 

(= )

The amplitudes are found easily

$$\mathcal{M}_{①} = -e^2\bar{u}_{s'}(\vec{p}\,')\,\not{\varepsilon}(\vec{k}')\cdot\frac{\not{p}+\not{k}+m}{(p+k)^2-m^2}\cdot\not{\varepsilon}(\vec{k})\,u_s(\vec{p}) \quad ,(8)$$

$$\mathcal{M}_{②} = -e^2 u_{s'}(\vec{p}\,')\,\not{\varepsilon}(\vec{k})\cdot\frac{\not{p}-\not{k}+m}{(p-k)^2-m^2}\cdot\not{\varepsilon}(\vec{k}')\,u_s(\vec{p}) \quad .(9)$$

Then

$$\mathcal{M} = \mathcal{M}_{①}+\mathcal{M}_{②} = -e^2\varepsilon_\mu(\vec{k}')\varepsilon_\nu(\vec{k})\,\bar{u}_{s'}(\vec{p}\,')\left(\frac{\gamma^\mu(\not{p}+\not{k}+m)\gamma^\nu}{(p+k)^2-m^2} + \frac{\gamma^\nu(\not{p}-\not{k}'+m)\gamma^\mu}{(p-k') -m^2}\right), \quad (10)$$

using the above & working in the laboratory frame, where the electron is initially at rest, we find

1. Scalar QED

The action reads

$$S = \int d^4x \left[-\frac{1}{4} F_{\mu\nu}^2 + \partial_\mu \varphi^* \partial_\mu \varphi - m^2 \varphi^* \varphi - ie A^\mu (\varphi \partial_\mu \varphi^* - \varphi^* \partial_\mu \varphi) + e^2 A_\mu A^\mu \varphi^* \varphi \right] \quad (1)$$

① We notice that we can rewrite the above as

$$S = \int d^4x \left[-\frac{1}{4} F_{\mu\nu}^2 + (D_\mu \varphi)^* D^\mu \varphi - m^2 \varphi^* \varphi \right], \quad (2)$$

with

$$D_\mu \varphi = \partial_\mu \varphi - ie A_\mu \varphi \quad , \quad (3)$$

the U(1) covariant derivative. Under

$$\varphi \to \varphi' = e^{ie\alpha} \varphi \,, \quad A_\mu \to A'_\mu = A_\mu + \partial_\mu \alpha \quad , \quad (4)$$

from eq. (3) we find that the covariant derivative transform as

$$(D_\mu \varphi)' = \partial_\mu \varphi' - ie A'_\mu \varphi' = e^{ie\alpha} D_\mu \varphi \quad . \quad (5)$$

Consequently,

$$(D_\mu \varphi)^{*\prime} = e^{-ie\alpha} (D_\mu \varphi)^* \quad , \quad (6)$$

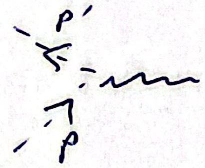
meaning that

$$(D_\mu \varphi)^{*\prime} (D^\mu \varphi)' = (D_\mu \varphi)^* (D^\mu \varphi) = \text{invariant} \, . \quad (7)$$

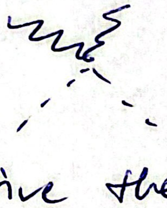
② The Feynman rules are :

- photon propagator : $= \dfrac{-i\eta_{\mu\nu}}{k^2}$

- scalar propagator : $= \dfrac{i}{p^2-m^2}$

- cubic vertex : 

$= ie(p_\mu+p'_\mu)$

- quartic vertex : 
$= 2ie^2\eta_{\mu\nu}$

The easiest way to derive the vertices is to go to momentum space by using

$$A^\mu(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ikx}\tilde{A}^\mu(k) \quad , \qquad (8)$$

$$\varphi(x) = \int \frac{d^4p}{(2\pi)^4} e^{-ipx}\tilde{\varphi}(p) \quad . \qquad (9)$$

The cubic part of the action is

$$S_{cubic} = -ie\int d^4x\, A^\mu(\varphi\partial_\mu\varphi^* - \varphi^*\partial_\mu\varphi) \ , \qquad (10)$$

which upon plugging (8),(9), it yields

$$S_{cubic} = -ie\int d^4x\int\frac{d^4k}{(2\pi)^4}\int\frac{d^4p}{(2\pi)^4}\int\frac{d^4p'}{(2\pi)^4}\tilde{A}^\mu(k)e^{-ikx}$$
$$[\tilde{\varphi}(p)e^{-ipx}\tilde{\varphi}^*(p')ip'_\mu e^{ip'x} - \tilde{\varphi}^*(p')e^{+ip'x}\tilde{\varphi}(p)(-ip_\mu)]$$
$$= e\int\frac{d^4k}{(2\pi)^4}\int\frac{d^4p}{(2\pi)^4}\int\frac{d^4p'}{(2\pi)^4}e^{-i(k+p-p')x}\tilde{A}^\mu(k)\tilde{\varphi}(p)\tilde{\varphi}^*(p')(p_\mu+p'_\mu) . \qquad (11)$$

Using the definition of the delta function, we get

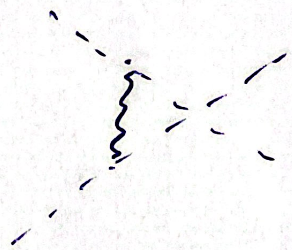
$$S_{cubic} = e\int\frac{d^4k}{(2\pi)^4}\int\frac{d^4p}{(2\pi)^4}\int\frac{d^4p'}{(2\pi)^4}(2\pi)^4\delta^{(4)}(k+p-p')\tilde{A}^\mu(k)\tilde{\varphi}(p)\tilde{\varphi}^*(p')(p_\mu+p'_\mu) . \qquad (12)$$

Integrating over k, we get

$$S_{\text{cubic}} = e\int \frac{d^4p}{(2\pi)^4}\int \frac{d^4p'}{(2\pi)^4}\, \tilde{A}^{\mu}(p-p')\tilde{\varphi}(p)\tilde{\varphi}^{*}(p')(p_{\mu}+p'_{\mu}) \quad ,(13)$$

from which we read-off the interaction vertex.

③,④ For $\varphi\phi \to \varphi\varphi$ scattering, we have the following diagrams

① =  , ② = 

with the corresponding amplitudes

$$\mathcal{M}_{①} = -e^2\frac{(p_1+p_3)_{\mu}(p_2+p_4)^{\mu}}{t} = -e^2\left(\frac{s-u}{t}\right), \quad (14)$$

$$\mathcal{M}_{②} = -e^2\frac{(p_1+p_4)_{\mu}(p_2+p_3)^{\mu}}{u} = -e^2\left(\frac{s-t}{u}\right). \quad (15)$$

Therefore, the total amplitude is

$$|\mathcal{M}|^2 = |\mathcal{M}_{①}+\mathcal{M}_{②}|^2 = e^4\left(\frac{s-u}{t}+\frac{s-t}{u}\right)^2, \quad (16)$$

translating into the following differential cross-section

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s}\left(\frac{s-u}{t}+\frac{s-t}{u}\right)^2 \quad ,(17)$$

where as usual

$$\alpha = \frac{e^2}{4\pi} \quad ,(18)$$

is the fine-structure constant.