Quantum Field Theory (Quantum Electrodynamics)

Problem Set 12 22 & 24 January 2024

1. Quantization of Maxwell theory

1. Let's now move to the quantization of the Maxwell theory, by promoting the a's into operators. Imposing canonical commutation relations between A_μ and $\dot{A}_\nu$, i.e.

$$[A_\mu(t,\vec{x}), A^\nu(t,\vec{x}')] = 0, \quad [\dot{A}_\mu(t,\vec{x}), A^\nu(t,\vec{x}')] = i\delta_\mu^\nu \delta^{(3)}(\vec{x}-\vec{x}') \ ,$$

show that the non-vanishing commutation relations between the ladder operators are

$$[a_r(\vec{k}), a_s^+(\vec{k}')] = 2\omega_{\vec{k}}(2\pi)^3 \zeta_s \delta_{rs} \delta^{(3)}(\vec{k}-\vec{k}') \ .$$

2. Argue why the Lorenz gauge condition cannot be imposed on operators, but should instead be imposed on states. Argue that fixing the gauge corresponds to

$$\Big(a_0(\vec{k}) - a_3(\vec{k})\Big) |\psi\rangle = 0 \ ,$$

(the Gupta–Bleuler condition), where $|\psi\rangle$ is a physical state.

3. Consider the state $a_0^+(\vec{k})\,|0\rangle$ and compute its norm. Is this state physical ? Modify this state accordingly, such that it satisfies the Gupta-Bleuler condition. Is it now physical ?
4. Express the Hamiltonian in terms of the ladder operators. Check that its expectation value in the modified states which you constructed in the previous point vanishes.
5. What statistics do vector particles obey ?
6. Compute the Feynman propagator for the vector field.

2. Quantum Electrodynamics (QED)

The Lagrangian of QED reads

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\bar{\psi}(\partial\!\!\!/ - m)\psi - e\bar{\psi}A\!\!\!/\psi \ , \tag{1}$$

with ψ a Dirac fermion with mass m.

1. Show that the Lagrangian is invariant under the gauge transformation $\psi \to \psi' = e^{-ie\alpha(x)}\psi,\ A_\mu \to A'_\mu = A_\mu + \partial_\mu \alpha(x)$, with $a(x)$ an arbitrary function.
2. The interaction part H_{int} of the Hamiltonian reads

$$H_{int} = e\int \mathrm{d}^3\vec{x}\,\bar{\psi}A\!\!\!/\psi \ .$$

Compute the matrix element

$$\langle f| \int \mathrm{d}t\, H_{int} |i\rangle \ ,$$

where the initial state comprises a fermion-antifermion pair, while the final state a photon. Can this process take place?

3. Let's add to (1) an additional Dirac fermion Ψ with mass M, such that

$$\tilde{\mathcal{L}} = \mathcal{L} + i\bar{\Psi}(\not{\partial} - M)\Psi - e\bar{\Psi}\not{A}\Psi \ , \tag{2}$$

where we may interpret ψ as an electron and Ψ as a muon. Compute, at the lowest nonvanishing order, the scattering amplitude for the process $\psi\Psi \to \psi\Psi$.

4. Square and average the above, write it in terms of the Mandelstam variables and derive the differential cross-section in the center-of-mass frame.

3. Feynman Rules

1. Work out the Feynman rules for the Lagrangian (2).
2. Using the above, compute the amplitude corresponding to the process $\psi\Psi \to \psi\Psi$ and compare your findings with Exercise 1, point 2.
3. Assuming that $m = 0$, compute the amplitude for the $\psi\psi \to \psi\psi$ scattering in terms of the Mandelstam variables. Derive the differential cross-section for this process.
4. Write the Feynman diagrams that contribute to the Compton scattering $\psi\gamma \to \psi\gamma$ and find the corresponding amplitude. Compute the differential cross section for this process in the laboratory frame.