1. Quantization of Maxwell theory

① We start from

$$[A_\mu(t,\vec{x}), A_\nu(t,\vec{x})] = 0 \quad , (1)$$

and use

$$A_\mu = \int d\tilde{k} \sum_{r=0}^{3} \varepsilon_{\mu,r} \left(a_r(\vec{k}) e^{-ikx} + a_r^\dagger(\vec{k}) e^{ikx} \right) , \quad (2)$$

where as usual

$$d\tilde{k} = \frac{d^3\vec{k}}{(2\pi)^3 2\omega_k} . \quad (3)$$

Then, we find

$$0 = \int d\tilde{k}\, d\tilde{k}' \sum_{r,s=0}^{3} \varepsilon_{\mu,r} \varepsilon_{\nu,s} \Big\{ [a_r(\vec{k}), a_s(\vec{k}')] e^{-ikx - ik'x'} + [a_r^\dagger(\vec{k}), a_s^\dagger(\vec{k}')] e^{ikx + ik'x'} \Big\} \quad (4)$$

For the above to hold, we have to impose that

$$[a_r(\vec{k}), a_s(\vec{k}')] = 0 , \quad [a_r^\dagger(\vec{k}), a_s^\dagger(\vec{k}')] = 0 . \quad (5)$$

We now move to the non-vanishing commutator

$$[A_\mu(t,\vec{x}), \dot{A}^\nu(t,\vec{x}')] = -i \delta_\mu^\nu \delta^{(3)}(\vec{x}-\vec{x}') . \quad (6)$$

From (2), we easily find

$$\dot{A}^\nu(t,\vec{x}') = -i \int d\tilde{k}\, \omega_{k'} \sum_{s=0} \varepsilon_s^\nu \left(a_s(\vec{k}) e^{-ik'x'} - a_s^\dagger(\vec{k}') e^{ik'x'} \right) , \quad (7)$$

which upon plugging into (6) yields

$$[A_\mu(t,\vec{x}), \dot{A}^\nu(t,\vec{x}')] = i \int d\tilde{k} \frac{d^3\vec{k}'}{(2\pi)^3} \sum_{s,r=0}^{3} \varepsilon_{\mu,r} \varepsilon_s^\nu [a_r(\vec{k}), a_s^\dagger(\vec{k}')] e^{-ikx + ik'x'} . \quad (8)$$

We notice that if

$$[a_r(\vec{k}), a_s^\dagger(\vec{k}')] = 2\omega_k (2\pi)^3 \zeta_s \delta_{rs} \delta^{(3)}(\vec{k}-\vec{k}') , \quad (9)$$

eq. (8) becomes

$$[A_\mu(t,\vec{x}), \dot{A}^\nu(t,\vec{x}')] = i \sum_{r=0}^{3} \zeta_r \varepsilon_{\mu,r} \varepsilon_s^\nu \int \frac{d^3\vec{k}}{(2\pi)^3} e^{i\vec{k}(\vec{x}-\vec{x}')} = i \delta_\mu^\nu \delta^{(3)}(\vec{x}-\vec{x}') . \quad (10)$$

②③. We cannot impose the Lorentz gauge condition on operators because it is in conflict with the canonical commutation relations.

Take the state $a_0^\dagger(\vec{k})|0\rangle$. Then, we have

$$\langle 0|a_0(\vec{k})a_0^\dagger(\vec{k})|0\rangle = -2\omega_k(2\pi)^3\delta^{(3)}(0) \quad , \quad (11)$$

where we used the commutation relations. The negative sign is obviously problematic, since it translates into a negative probability.

To remedy the situation, we have to get rid of states with negative norm. To this end, we introduce a "physical state" $|\psi\rangle$, that contains a timelike and a longitudinal photon, i.e.

$$|\psi\rangle : \ (a_0(\vec{k}) - a_3(\vec{k}))|\psi\rangle = 0 \quad . \quad (12)$$

This implies that

$$\langle\psi|(a_0^\dagger(\vec{k}) - a_3^\dagger(\vec{k})) = 0 \quad , \quad (13)$$

from which we find

$$\langle\psi|(a_0^\dagger a_0 - a_3^\dagger a_3)|\psi\rangle = 0 \ . \ (14)$$

④ The normal-ordered Hamiltonian reads

$$H = \int \frac{d^3\vec{k}}{2(2\pi)^3} \sum_{r=0}^{3} \zeta_r \, a_r^\dagger(\vec{k})a_r(\vec{k}) \quad . \quad (15)$$

We sandwich the above between states subject to (14) to find

$$\langle\psi|H|\psi\rangle = \langle\psi|\int\frac{d^3\vec{k}}{2(2\pi)^3}\sum_{r=1,2} a_r^\dagger(\vec{k})a_r(\vec{k})|\psi\rangle \qquad (16)$$

$$+ \langle\psi|\int\frac{d^3\vec{k}}{2(2\pi)^3}(-a_0^\dagger(\vec{k})a_0(\vec{k}) + a_3^\dagger(\vec{k})a_3(\vec{k}))|\psi\rangle .$$

Owing to (14), the second line identically vanishes, meaning that the expectation value of the Hamiltonian on physical states only contains the transverse

⑤ Consider the state

$$a_r^\dagger(\vec{k})a_s^\dagger(\vec{k}')|0\rangle \quad . \tag{17}$$

Since vector particles commute, the above gives

$$a_r^\dagger(\vec{k})a_s^\dagger(\vec{k}')|0\rangle = a_s^\dagger(\vec{k}')a_r^\dagger(\vec{k})|0\rangle \Rightarrow \text{Bose-Einstein}. \tag{18}$$

⑥ To find the Feynman propagator we first compute the Wightman function for the vector field

$$D^{\mu\nu}(x-x') = \langle 0|A^\mu(x)A^\nu(x')|0\rangle \quad . \tag{19}$$

Using (2), the above becomes

$$D^{\mu\nu}(x-x') = \int d\tilde{k}\,d\tilde{k}' \sum_{r,s=0}^{3} \varepsilon_r^\mu(\vec{k})\varepsilon_s^\nu(\vec{k}')\langle 0|a_r(\vec{k})a_s^\dagger(\vec{k}')|0\rangle e^{-ikx+ik'x'}$$

$$= \int d\tilde{k}\,d\tilde{k}' \sum_{r,s=0}^{3} \varepsilon_r^\mu \varepsilon_s^\nu(\vec{k}')\,2\omega_{k'}(2\pi)^3 \zeta_r \delta_{rs}\,\delta^{(3)}(\vec{k}-\vec{k}')e^{-ikx+ik'x'}$$

$$= \int d\tilde{k} \sum_{r=0}^{3} \zeta_r \varepsilon_r^\mu(\vec{k})\varepsilon_r^\nu(\vec{k})\, e^{-ik(x-x')} ,$$

$$\rightarrow D^{\mu\nu}(x-x') = -\eta^{\mu\nu}\int d\tilde{k}\, e^{-ik(x-x')} \quad . \tag{20}$$

From the Wightman function, we can readily compute the Feynman propagator

$$iD_F^{\mu\nu}(x-x') = \theta(t-t')D^{\mu\nu}(x-x') + \theta(t'-t)D^{\mu\nu}(x'-x) \quad . \tag{21}$$

Using the results of Problem Set 8, we get

$$D_F^{\mu\nu}(x-x') = \lim_{\epsilon\to 0}\int \frac{d^4k}{(2\pi)^4}\, e^{-ik(x-x')}\,\frac{-\eta^{\mu\nu}}{k^2+i\epsilon} \quad . \tag{22}$$

2. Quantum Electrodynamics (QED)

① The QED Lagrangian reads

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 + i\bar{\psi}(\not{\partial} - m)\psi - e\bar{\psi}\not{A}\psi \quad . (1)$$

Consider the following transformation

$$\psi \to \psi' = e^{-ie\alpha}\psi \quad \rightsquigarrow \quad \bar{\psi} \to \bar{\psi}' = e^{ie\alpha}\bar{\psi} \quad , A_\mu \to A'_\mu = A_\mu + \partial_\mu \alpha \, . (2)$$

The infinitesimal forms of the above are

$$\delta\psi = -ie\alpha\psi \, , \, \delta\bar{\psi} = ie\alpha\bar{\psi} \, , \, \delta A_\mu = \partial_\mu \alpha \quad . (3)$$

The response of the Lagrangian (1) under (3) is

$$\delta\mathcal{L} = -\frac{1}{2}F_{\mu\nu}\delta F^{\mu\nu} + i\,\delta\bar{\psi}(\not{\partial} - m)\psi + i\bar{\psi}(\not{\partial} - m)\delta\psi - e\,\delta\bar{\psi}\not{A}\psi$$
$$- e\bar{\psi}\delta\not{A}\psi - e\bar{\psi}\not{A}\delta\psi$$
$$= -e\alpha\bar{\psi}(\not{\partial} - m) + e\bar{\psi}(\not{\partial} - m)(\alpha\psi) \cancel{-ie^2\alpha\bar{\psi}\not{A}\psi} - e\bar{\psi}(\not{\partial}\alpha)\psi$$
$$\cancel{+ie^2\alpha\bar{\psi}\not{A}\psi}$$
$$= \cancel{-e\alpha\bar{\psi}(\not{\partial} - m)\psi} \cancel{+ e\bar{\psi}(\not{\partial} - m)\psi} + e\bar{\psi}(\not{\partial}\alpha)\psi - e\bar{\psi}(\not{\partial}\alpha)\psi = 0$$

$$\Rightarrow \delta\mathcal{L} = 0 \quad , \quad (4)$$

showing that $\mathcal{L}$ is indeed invariant under (3).

② The interaction part of the Hamiltonian is

$$H_{int} = e\int d^3\vec{x}\, \bar{\psi}\not{A}\psi \quad . (5)$$

The mode expansion for fermions & the vector field are

$$\psi = \int d\tilde{p} \sum_i \left(u_i(\vec{p})a_i(\vec{p})e^{-ipx} + v_i(\vec{p})b_i^\dagger(\vec{p})e^{ipx}\right) \quad , (6)$$

$$A^\mu = \int d\tilde{k} \sum_s \varepsilon_s^\mu \left(c_s(\vec{k})e^{-ikx} + c_s^\dagger(\vec{k})e^{ikx}\right) \quad . (7)$$

Take an initial state comprising a fermion-antifermion pair

$$|i\rangle = a_{s_1}^\dagger(\vec{p})b_{s_2}^\dagger(\vec{p}')|0\rangle \quad , \quad (8)$$

and a final state comprising a single photon

$$|f\rangle = c^\dagger(\vec{k})|0\rangle \quad . (9)$$

Then, the matrix element is

$$\langle f|\int dt\, \mathcal{H}_{int}|i\rangle = e\int d^4x \langle 0|c_r(\vec{k})\bar{\psi}\cancel{A}\psi a^\dagger_{s_1}(\vec{p})b^\dagger_{s_2}(\vec{p}\,')|0\rangle \;. \quad (10)$$

The only non-zero contraction of (10) is

$$\langle f|\int dt\, \mathcal{H}_{int}|i\rangle = e\int d^4x \langle 0|c_r(\vec{k})\bar{\psi}\cancel{A}\psi a^\dagger_{s_1}(\vec{p})b^\dagger_{s_2}(\vec{p}\,')|0\rangle \;. \quad (11)$$

using

$$\underbrace{c_r(\vec{k})A^\mu} = \varepsilon^\mu_r e^{ikx} \;, \quad (12)$$

$$\underbrace{\bar{\psi}\, b^\dagger_{s_2}(\vec{p}\,')} = \bar{v}_{s_2}(\vec{p}\,')e^{-ip'x} \;, \quad (13)$$

$$\psi a^\dagger_{s_1}(\vec{p}) = u_{s_1}(\vec{p})e^{-ipx} \;, \quad (14)$$

as follows from (6), (7), we obtain

$$\langle f|\int dt\, \mathcal{H}_{int}|i\rangle = e(2\pi)^4\delta^{(4)}(k-p-p')\bar{v}_{s_2}(\vec{p}\,')\gamma^\mu\varepsilon_{\mu,r}u_{s_1}(\vec{p}) \;. \quad (15)$$

Go now to a frame where $\vec{p}+\vec{p}\,'=0$. From the above we notice that momentum conservation dictates that the photon in the final state also have vanishing three-momentum, i.e. $\vec{k}=0$. This is of course not kinematically allowed.

③ We now add to the Lagrangian of QED another piece containing an additional fermion Ψ:

$$\tilde{\mathcal{L}} = \mathcal{L} + i\bar{\Psi}(\cancel{\partial}-M) - e\bar{\Psi}\cancel{A}\Psi \;. \quad (16)$$

At the lowest non-vanishing order, the associated amplitude reads

$$e^2\langle f|\int d^4x\int d^4x'\, T\{\bar{\psi}(x)\cancel{A}(x)\psi(x)\bar{\Psi}(x')\cancel{A}(x')\Psi(x')\}|i\rangle \;, \quad (17)$$

with the initial & final states given by

$$|i\rangle = a^\dagger_{s_1}(\vec{p}_1)\alpha^\dagger_{s_2}(\vec{p}_2)|0\rangle \;, \quad |f\rangle = a^\dagger_{s_1}(\vec{p}_3)\alpha^\dagger_{s_4}(\vec{p}_4)|0\rangle \;, \quad (18)$$

~~where~~ $a_\#$, $\alpha_\#$ correspond to ψ, Ψ, respectively.

The only non-vanishing contraction of the above is

$$e^2\int d^4x\int d^4x' \langle 0|a_{s_3}(\vec{p}_3)\alpha_{s_4}(\vec{p}_4)\bar{\psi}(x)\not{A}(x)\psi(x)\bar{\Psi}(x')\not{A}(x')\Psi(x')a^{+}_{s_1}(\vec{p}_1)\alpha^{+}_{s_2}(\vec{p}_2)|0\rangle \tag{19}$$

We now use

$$a_{s_3}(\vec{p}_3)\bar{\psi}(x) = \bar{u}^{\psi}_{s_3}(\vec{p}_3)e^{ip_3x} \quad , \tag{20}$$

$$A^{\mu}(x)A^{\nu}(x') = \int\frac{d^4k}{(2\pi)^4}e^{-ik(x-x')}\frac{-i\eta^{\mu\nu}}{k^2} \quad , \tag{21}$$

$$\psi(x)a^{+}_{s_1}(\vec{p}_1) = u^{\psi}_{s_1}(\vec{p}_1)e^{-ip_1x} \quad , \tag{22}$$

$$\alpha_{s_4}(\vec{p}_4)\bar{\Psi}(x') = \bar{u}^{\Psi}_{s_4}(\vec{p}_4)e^{ip_4x'} \quad , \tag{23}$$

$$\Psi(x')\alpha^{+}_{s_2}(\vec{p}_2) = u^{\Psi}_{s_2}(\vec{p}_2)e^{-ip_2x'} \quad , \tag{24}$$

where the superscript in the spinors was introduced to distinguish between ψ, Ψ. Plugging (20) - (24) into (19), we obtain

$$-ie^2(2\pi)^4\delta^{(4)}(p_1+p_2-p_3-p_4)\,\bar{u}^{\psi}_{s_3}(\vec{p}_3)\gamma^{\mu}u^{\psi}_{s_1}(\vec{p}_1)\bar{u}^{\Psi}_{s_4}(\vec{p}_4)\gamma_{\mu}u^{\Psi}_{s_2}(\vec{p}_2)\frac{1}{(p_1-p_3)^2} \quad . \tag{25}$$

The corresponding amplitude is then

$$\mathcal{M} = \frac{e^2}{t}\bar{u}^{\psi}_{s_3}(\vec{p}_3)\gamma^{\mu}u^{\psi}_{s_1}(\vec{p}_1)\bar{u}^{\Psi}_{s_4}(\vec{p}_4)\gamma_{\mu}u^{\Psi}_{s_2}(\vec{p}_2) \quad , \tag{26}$$

with

$$t = (p_1-p_3)^2 \quad , \tag{27}$$

the Mandelstam variable.

④ We have to compute

$$|\mathcal{M}|^2 = \frac{e^4}{t^2} M^\psi_{\mu\nu} M^{\Psi\mu\nu} \quad , \qquad (28)$$

with

$$M^\psi_{\mu\nu} = \bar u^\psi_{s_3}(\vec p_3)\gamma_\mu u^\psi_{s_1}(\vec p_1)\bar u^\psi_{s_1}(\vec p_1)\gamma_\nu u^\psi_{s_3}(\vec p_3) \quad , \qquad (29)$$

$$M^\Psi_{\mu\nu} = \bar u^\Psi_{s_4}(\vec p_4)\gamma_\mu u^\Psi_{s_2}(\vec p_2)\bar u^\Psi_{s_2}(\vec p_2)\gamma_\nu u^\Psi_{s_4}(\vec p_4) \quad . \qquad (30)$$

Therefore,

$$|\bar{\mathcal{M}}^2| = \frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{e^4}{4t^2} \sum_{s_1,s_3} M^\psi_{\mu\nu} \sum_{s_2,s_4} M^{\Psi\mu\nu} \quad . \qquad (31)$$

Let's compute the above in details

$$⊛ \sum_{s_1,s_3} M^\psi_{\mu\nu} = \sum_{s_1,s_3} \bar u^\psi_{s_3}(\vec p_3)^{\alpha'} \gamma_\mu^{\alpha'\beta'} u^\psi_{s_1}(\vec p_1)^{\beta'} \bar u^\psi_{s_1}(\vec p_1)^{\gamma'} \gamma_\nu^{\gamma'\delta'} u^\psi_{s_3}(\vec p_3)^{\delta'} \quad , \qquad (32)$$

where we explicitly wrote down the spinor indexes α', β'...
we find

$$\sum_{s_1,s_3} M^\psi_{\mu\nu} = \sum_{s_3} \bar u^\psi_{s_3}(\vec p_3)^{\alpha'} \gamma_\mu^{\alpha'\beta'} (\not p_1 + m)^{\beta'\gamma'} \gamma_\nu^{\gamma'\delta'} u^\psi_{s_3}(\vec p_3)^{\delta'}$$

$$= (\not p_3 + m)^{\delta'\alpha'} \gamma_\mu^{\alpha'\beta'} (\not p_1 + m)^{\beta'\gamma'} \gamma_\nu^{\gamma'\delta'}$$

$$= \mathrm{tr}\left[(\not p_3 + m)\gamma_\mu(\not p_1 + m)\gamma_\nu\right] = \mathrm{tr}\Big[\not p_3\gamma_\mu\not p_1\gamma_\nu + \overbrace{\cancelto{0}{\not p_3\gamma_\mu m\gamma_\nu} + \cancelto{0}{m\gamma_\mu\not p_1\gamma_\nu}}^{\text{odd \# of } \gamma\text{'s}}$$

$$+ m^2\gamma_\mu\gamma_\nu\Big]$$

$$= \mathrm{tr}\left[\not p_3\gamma_\mu\not p_1\gamma_\nu + m^2\gamma_\mu\gamma_\nu\right] = \mathrm{tr}\left[\gamma_\alpha\gamma_\mu\gamma_\beta\gamma_\nu p_3^\alpha p_1^\beta + m^2\gamma_\mu\gamma_\nu\right]$$

$$\rightarrow \sum_{s_1,s_3} M^\psi_{\mu\nu} = 4\left(p_{1\nu}p_{3\mu} + p_{1\mu}p_{3\nu} - \eta_{\mu\nu}(p_1\cdot p_3 - m^2)\right) \quad . \qquad (33)$$

Similarly, we obtain

$$⊛ \sum_{s_2,s_4} M^\Psi_{\mu\nu} = 4\left(p_{2\nu}p_{4\mu} + p_{2\mu}p_{4\nu} - \eta_{\mu\nu}(p_2\cdot p_4 - M^2)\right) \quad . \qquad (34)$$

Therefore, after some algebra we obtain

$$|\bar{M}|^2 = \frac{8e^4}{t^2}\left(\frac{s^2+u^2}{4} - \frac{1}{2}(m^2+M^2)^2 + t(m^2+M^2)\right), \quad (35)$$

where

$$s = (p_1+p_2)^2, \qquad u = (p_1-p_4)^2 . \quad (36)$$

The differential cross section reads

$$\frac{d\sigma}{d\Omega}(\psi\Psi \to \psi\Psi) = \frac{|\bar{M}|^2}{64\pi^2 s} = \frac{\alpha^2}{2s}\left(\frac{s^2+u^2-2(m^2+M^2)^2}{t^2} + \frac{4(m^2+M^2)}{t}\right), \quad (37)$$

with

$$\alpha = \frac{e^2}{4\pi} = \text{fine-structure constant} . \quad (38)$$