$$|i\rangle = a^{\dagger}_{s_1}(\vec{p}_1)\, a^{\dagger}_{s_2}(\vec{p}_2)|0\rangle \;;\; |f\rangle = a^{\dagger}_{s_3}(\vec{p}_3)\, a^{\dagger}_{s_4}(\vec{p}_4)|0\rangle$$

$\leftarrow$ both don't contain $c^{\dagger}(\vec{k})$

(1) $\langle f|(-i)\int_{-\infty}^{+\infty} dt\, T H_{int}|i\rangle = -ig\int d^4x \langle f| T\{\bar{\psi}(x)\psi(x)\varphi(x)\}|i\rangle = 0$

since by Wick's Theorem only a fully contracted set of fields can give rise to a nonzero vacuum expectation value. An odd number of fields can not be fully contracted as each Wick contraction involves a pair of fields. Also, $[a^{(\dagger)}, c^{(\dagger)}] = [b^{(\dagger)}, c^{(\dagger)}] = 0$ & $c|0\rangle = 0$.

Reminder: <u>Wick's Theorem:</u> $T\{\varphi(x_1)\ldots\varphi(x_n)\} = N\{\varphi(x_1)\ldots\varphi(x_n) + \text{all possible contractions}\}$

also: $\langle 0|N(\text{any operator})|0\rangle = 0$

(2) $T\{\bar{\psi}(x)\psi(x)\varphi(x)\bar{\psi}(x')\psi(x')\varphi(x')\} = :\bar{\psi}(x)\psi(x)\bar{\psi}(x')\psi(x'): \underbrace{\varphi(x)\varphi(x')}_{= D_F(x,x')}$

since applying Wick's Theorem we obtain a non-vanishing result only if $\varphi(x)$ is Wick-contracted with $\varphi(x')$.

also: $|i\rangle, |f\rangle$ both don't contain $c^{\dagger}(\vec{k})$ $\Rightarrow$ can't contract w/ $\varphi(x), \varphi(x')$

(3) $I \equiv \int_{x,x'} \langle f | : \bar{\psi}(x)\psi(x)\bar{\psi}(x')\psi(x') : | i \rangle \mathcal{D}_F(x,x') \quad ; \int_{x,x'} \equiv \int d^4x \int d^4x'$

$\psi(x) = \int_p \sum_i \left[u_i(\vec{p})\, a_i(\vec{p})\, e^{-ip\cdot x} + v_i(\vec{p})\, b_i^\dagger(\vec{p})\, e^{ip\cdot x} \right] \quad ; \int_p \equiv \int \frac{d^3\vec{p}}{(2\pi)^3 2\omega_{\vec{p}}}$

Only the terms with $a, a^\dagger$ contribute to I;
physically, because of $|i\rangle$ & $|f\rangle$, we need 2 a's / $a^\dagger$'s to annihilate / create the corresponding number of particles to get from $|i\rangle$ to $|f\rangle$.

$$\Rightarrow I = \int_{x,x',p,q,p',q'} \sum_{i,j,k,l} \langle f | : \bar{u}_i(\vec{p})\, a_i^\dagger(\vec{p})\, e^{ipx}\, u_j(\vec{q})\, a_j(\vec{q})\, e^{-iqx}\, \bar{u}_k(\vec{p}')\, a_k^\dagger(\vec{p}')\, e^{ip'x'} \cdot$$

$$\cdot\, u_l(\vec{q}')\, a_l(\vec{q}')\, e^{-iq'x'} : | i \rangle \mathcal{D}_F(x,x') =$$

$$= -\int_{x,x',p,q,p',q'} \sum_{i,j,k,l} \bar{u}_i(\vec{p})\, u_j(\vec{q})\, \bar{u}_k(\vec{p}')\, u_l(\vec{q}')\, e^{ipx} e^{-iqx} e^{ip'x'} e^{-iq'x'} \langle 0 | a_{s_4}(\vec{p}_4) \overbrace{a_{s_3}(\vec{p}_3)\, a_i^\dagger(\vec{p})}^{= -a_i^\dagger(\vec{p})\, a_{s_3}(\vec{p}_3) + (2\pi)^3 2\omega_{\vec{p}}\, \delta_{i,s_3}\, \delta^{(3)}(\vec{p}-\vec{p}_3)} \cdot$$

$$\cdot\, a_k^\dagger(\vec{p}')\, a_j(\vec{q})\, a_l(\vec{q}')\, a_{s_1}^\dagger(\vec{p}_1)\, a_{s_2}^\dagger(\vec{p}_2) | 0 \rangle \mathcal{D}_F(x,x')$$

$$\langle 0 | \ldots | 0 \rangle = \langle 0 | \Big[- \underbrace{a_{s_4}(\vec{p}_4)\, a_i^\dagger(\vec{p})}_{= -a_i^\dagger(\vec{p})\, a_{s_4}(\vec{p}_4) + (2\pi)^3 2\omega_{\vec{p}}\, \delta_{i,s_4}\, \delta^{(3)}(\vec{p}-\vec{p}_4)} \overbrace{a_{s_3}(\vec{p}_3)\, a_k^\dagger(\vec{p}')}^{= -a_k^\dagger(\vec{p}')\, a_{s_3}(\vec{p}_3) + (2\pi)^3 2\omega_{\vec{p}'}\, \delta_{k,s_3}\, \delta^{(3)}(\vec{p}'-\vec{p}_3)} + (2\pi)^3 2\omega_{\vec{p}}\, \delta_{i,s_3}\, \delta^{(3)}(\vec{p}-\vec{p}_3) \overbrace{a_{s_4}(\vec{p}_4)\, a_k^\dagger(\vec{p}')}^{= -a_k^\dagger(\vec{p}')\, a_{s_4}(\vec{p}_4) + (2\pi)^3 2\omega_{\vec{p}'}\, \delta_{k,s_4}\, \delta^{(3)}(\vec{p}'-\vec{p}_4)} \Big] a_j(\vec{q}) \underbrace{a_l(\vec{q}')\, a_{s_1}^\dagger(\vec{p}_1)}_{= -a_{s_1}^\dagger(\vec{p}_1)\, a_l(\vec{q}') + (2\pi)^3 2\omega_{\vec{q}'}\, \delta_{l,s_1}\, \delta^{(3)}(\vec{q}'-\vec{p}_1)} a_{s_2}^\dagger(\vec{p}_2) | 0 \rangle =$$

$\langle 0 | a_i^\dagger(\vec{p}) = \langle 0 | a_k^\dagger(\vec{p}') = 0$

$$= \langle 0 | \Big[-(2\pi)^3 2\omega_{\vec{p}}\, \delta_{i,s_4}\, \delta^{(3)}(\vec{p}-\vec{p}_4)\, (2\pi)^3 2\omega_{\vec{p}'}\, \delta_{k,s_3}\, \delta^{(3)}(\vec{p}'-\vec{p}_3) +$$

$$+ (2\pi)^3 2\omega_{\vec{p}}\, \delta_{i,s_3}\, \delta^{(3)}(\vec{p}-\vec{p}_3)\, (2\pi)^3 2\omega_{\vec{p}'}\, \delta_{k,s_4}\, \delta^{(3)}(\vec{p}'-\vec{p}_4) \Big] \cdot$$

$$\cdot \Big[- \underbrace{a_j(\vec{q})\, a_{s_1}^\dagger(\vec{p}_1)}_{= -a_{s_1}^\dagger(\vec{p}_1)\, a_j(\vec{q}) + (2\pi)^3 2\omega_{\vec{q}}\, \delta_{j,s_1}\, \delta^{(3)}(\vec{q}-\vec{p}_1)} \underbrace{a_l(\vec{q}')\, a_{s_2}^\dagger(\vec{p}_2)}_{= -a_{s_2}^\dagger(\vec{p}_2)\, a_l(\vec{q}') + (2\pi)^3 2\omega_{\vec{q}'}\, \delta_{l,s_2}\, \delta^{(3)}(\vec{q}'-\vec{p}_2)} + (2\pi)^3 2\omega_{\vec{q}'}\, \delta_{l,s_1}\, \delta^{(3)}(\vec{q}'-\vec{p}_1) \underbrace{a_j(\vec{q})\, a_{s_2}^\dagger(\vec{p}_2)}_{= -a_{s_2}^\dagger(\vec{p}_2)\, a_j(\vec{q}) + (2\pi)^3 2\omega_{\vec{q}}\, \delta_{j,s_2}\, \delta^{(3)}(\vec{q}-\vec{p}_2)} \Big] | 0 \rangle =$$

$\langle 0 | a_{s_1}^\dagger(\vec{p}_1) = \langle 0 | a_{s_2}^\dagger(\vec{p}_2) = 0$

$$= \Big[-(2\pi)^3 2\omega_{\vec{p}}\, \delta_{i,s_4}\, \delta^{(3)}(\vec{p}-\vec{p}_4)\, (2\pi)^3 2\omega_{\vec{p}'}\, \delta_{k,s_3}\, \delta^{(3)}(\vec{p}'-\vec{p}_3) +$$

$$+ (2\pi)^3 2\omega_{\vec{p}}\, \delta_{i,s_3}\, \delta^{(3)}(\vec{p}-\vec{p}_3)\, (2\pi)^3 2\omega_{\vec{p}'}\, \delta_{k,s_4}\, \delta^{(3)}(\vec{p}'-\vec{p}_4) \Big] \cdot$$

$$\cdot \Big[-(2\pi)^3 2\omega_{\vec{q}}\, \delta_{j,s_1}\, \delta^{(3)}(\vec{q}-\vec{p}_1)\, (2\pi)^3 2\omega_{\vec{q}'}\, \delta_{l,s_2}\, \delta^{(3)}(\vec{q}'-\vec{p}_2) +$$

$$+ (2\pi)^3 2\omega_{\vec{q}}\, \delta_{j,s_2}\, \delta^{(3)}(\vec{q}-\vec{p}_2)\, (2\pi)^3 2\omega_{\vec{q}'}\, \delta_{l,s_1}\, \delta^{(3)}(\vec{q}'-\vec{p}_1) \Big]$$

Integrate I over p, p':

$$I=\int_{x,x',q,q'}\sum_{j,l}\Big(\bar{u}_{s_4}(\vec{p}_4)e^{ip_4x}u_j(\vec{q})e^{-iqx}\bar{u}_{s_3}(\vec{p}_3)e^{ip_3x'}u_l(\vec{q}\,')e^{-iq'x'}-$$

$$-\bar{u}_{s_3}(\vec{p}_3)e^{ip_3x}u_j(\vec{q})e^{-iqx}\bar{u}_{s_4}(\vec{p}_4)e^{ip_4x'}u_l(\vec{q}\,')e^{-iq'x'}\Big)\cdot$$

$$\cdot\Big[(2\pi)^3 2\omega_{\vec{q}}\,\delta_{j,s_2}\delta^{(3)}(\vec{q}-\vec{p}_2)(2\pi)^3 2\omega_{\vec{q}\,'}\delta_{l,s_1}\delta^{(3)}(\vec{q}\,'-\vec{p}_1)-$$

$$-(2\pi)^3 2\omega_{\vec{q}}\,\delta_{j,s_1}\delta^{(3)}(\vec{q}-\vec{p}_1)(2\pi)^3 2\omega_{\vec{q}\,'}\delta_{l,s_2}\delta^{(3)}(\vec{q}\,'-\vec{p}_2)\Big]\mathcal{D}_F(x,x')$$

Integrate I over q,q':

$$I=\int_{x,x'}\Big(\bar{u}_{s_4}(\vec{p}_4)u_{s_2}(\vec{p}_2)\bar{u}_{s_3}(\vec{p}_3)u_{s_1}(\vec{p}_1)e^{ip_4x}e^{-ip_2x}e^{ip_3x'}e^{-ip_1x'}-$$

$$-\bar{u}_{s_4}(\vec{p}_4)u_{s_1}(\vec{p}_1)\bar{u}_{s_3}(\vec{p}_3)u_{s_2}(\vec{p}_2)e^{ip_4x}e^{-ip_1x}e^{ip_3x'}e^{-ip_2x'}-$$

$$-\bar{u}_{s_3}(\vec{p}_3)u_{s_2}(\vec{p}_2)\bar{u}_{s_4}(\vec{p}_4)u_{s_1}(\vec{p}_1)e^{ip_3x}e^{-ip_2x}e^{ip_4x'}e^{-ip_1x'}+$$

$$+\bar{u}_{s_3}(\vec{p}_3)u_{s_1}(\vec{p}_1)\bar{u}_{s_4}(\vec{p}_4)u_{s_2}(\vec{p}_2)e^{ip_3x}e^{-ip_1x}e^{ip_4x'}e^{-ip_2x'}\Big)\overbrace{\mathcal{D}_F(x,x')}^{=\mathcal{D}_F(x',x)\text{ i.e. symmetric in exchange of }x,x'}=$$

$$=2\int_{x,x'}\Big(\bar{u}_{s_4}(\vec{p}_4)u_{s_2}(\vec{p}_2)\bar{u}_{s_3}(\vec{p}_3)u_{s_1}(\vec{p}_1)e^{ip_4x}e^{-ip_2x}e^{ip_3x'}e^{-ip_1x'}-$$

$$-\bar{u}_{s_4}(\vec{p}_4)u_{s_1}(\vec{p}_1)\bar{u}_{s_3}(\vec{p}_3)u_{s_2}(\vec{p}_2)e^{ip_4x}e^{-ip_1x}e^{ip_3x'}e^{-ip_2x'}\Big)\mathcal{D}_F(x,x')$$

Reminder: $\mathcal{D}_F(x,x')=\int\frac{d^4p}{(2\pi)^4}\frac{i\,e^{-ip(x-x')}}{p^2-M^2}$

$$I=2(2\pi)^4\int\frac{d^4p\;i}{p^2-M^2}\Big(\delta^{(4)}(p_4-p_2-p)\delta^{(4)}(p_3-p_1+p)\bar{u}_{s_4}(\vec{p}_4)u_{s_2}(\vec{p}_2)\bar{u}_{s_3}(\vec{p}_3)u_{s_1}(\vec{p}_1)-$$

$$-\delta^{(4)}(p_4-p_1-p)\delta^{(4)}(p_3-p_2+p)\bar{u}_{s_4}(\vec{p}_4)u_{s_1}(\vec{p}_1)\bar{u}_{s_3}(\vec{p}_3)u_{s_2}(\vec{p}_2)\Big)=$$

$$=(2\pi)^4\delta^{(4)}(p_3+p_4-p_1-p_2)\,i\,2\left(\frac{\bar{u}_{s_4}(\vec{p}_4)u_{s_2}(\vec{p}_2)\bar{u}_{s_3}(\vec{p}_3)u_{s_1}(\vec{p}_1)}{(p_1-p_3)^2-M^2}-\frac{\bar{u}_{s_4}(\vec{p}_4)u_{s_1}(\vec{p}_1)\bar{u}_{s_3}(\vec{p}_3)u_{s_2}(\vec{p}_2)}{(p_1-p_4)^2-M^2}\right)$$

From $(2\pi)^4\delta^{(4)}(p_3+p_4-p_1-p_2)\,i\mathcal{M}=-\frac{g^2}{2}I$ & $t=(p_1-p_3)^2$, $u=(p_1-p_4)^2$:

$$\mathcal{M}=-g^2\left(\frac{\bar{u}_{s_4}(\vec{p}_4)u_{s_2}(\vec{p}_2)\bar{u}_{s_3}(\vec{p}_3)u_{s_1}(\vec{p}_1)}{t-M^2}-\frac{\bar{u}_{s_4}(\vec{p}_4)u_{s_1}(\vec{p}_1)\bar{u}_{s_3}(\vec{p}_3)u_{s_2}(\vec{p}_2)}{u-M^2}\right)\equiv\mathcal{M}_t+\mathcal{M}_u$$

(4) $\overline{|\mathcal{M}|^2} = \frac{1}{4}\sum_{spins} |\mathcal{M}|^2 = \frac{1}{4}\sum_{spins}\Big(\underbrace{|\mathcal{M}_t|^2}_{①} + \underbrace{|\mathcal{M}_u|^2}_{②} + \underbrace{\mathcal{M}_t\mathcal{M}_u^\dagger + \mathcal{M}_u\mathcal{M}_t^\dagger}_{③}\Big)$

① $\frac{1}{4}\sum_{spins} |\mathcal{M}_t|^2 = \frac{g^4}{4(t-M^2)^2}\sum_{spins} \bar{u}_{s_{4\alpha}}(\vec{p}_4)\, u_{s_{2\alpha}}(\vec{p}_2)\, \bar{u}_{s_{3\beta}}(\vec{p}_3)\, u_{s_{1\beta}}(\vec{p}_1)\cdot \bar{u}_{s_{1\gamma}}(\vec{p}_1)\, u_{s_{3\gamma}}(\vec{p}_3)\, \bar{u}_{s_{2\delta}}(\vec{p}_2)\, u_{s_{4\delta}}(\vec{p}_4) =$

$= \frac{g^4}{4(t-M^2)^2}\, tr\left[(\not{p}_1+m)(\not{p}_3+m)\right] tr\left[(\not{p}_2+m)(\not{p}_4+m)\right] =$

$= \frac{4g^4}{(t-M^2)^2}(p_1\cdot p_3 + m^2)(p_2\cdot p_4 + m^2)$

② $\frac{1}{4}\sum_{spins} |\mathcal{M}_u|^2 = \frac{g^4}{4(u-M^2)^2}\sum_{spins} \bar{u}_{s_4}(\vec{p}_4)\, u_{s_1}(\vec{p}_1)\, \bar{u}_{s_3}(\vec{p}_3)\, u_{s_2}(\vec{p}_2)\cdot \bar{u}_{s_2}(\vec{p}_2)\, u_{s_3}(\vec{p}_3)\, \bar{u}_{s_1}(\vec{p}_1)\, u_{s_4}(\vec{p}_4) =$

$= \frac{g^4}{4(u-M^2)^2}\, tr\left[(\not{p}_2+m)(\not{p}_3+m)\right] tr\left[(\not{p}_1+m)(\not{p}_4+m)\right] = \frac{4g^4}{(u-M^2)^2}(p_2\cdot p_3 + m^2)(p_1\cdot p_4 + m^2)$

③ $\frac{1}{4}\sum_{spins}(\mathcal{M}_t\mathcal{M}_u^\dagger + \mathcal{M}_u\mathcal{M}_t^\dagger) = \frac{-g^4}{4(t-M^2)(u-M^2)}\,\cdot$

$\cdot \sum_{spins}\Big(\bar{u}_{s_4}(\vec{p}_4)\, u_{s_2}(\vec{p}_2)\, \bar{u}_{s_3}(\vec{p}_3)\, u_{s_1}(\vec{p}_1)\, \bar{u}_{s_2}(\vec{p}_2)\, u_{s_3}(\vec{p}_3)\, \bar{u}_{s_1}(\vec{p}_1)\, u_{s_4}(\vec{p}_4) +$

$+ \bar{u}_{s_4}(\vec{p}_4)\, u_{s_1}(\vec{p}_1)\, \bar{u}_{s_3}(\vec{p}_3)\, u_{s_2}(\vec{p}_2)\, \bar{u}_{s_1}(\vec{p}_1)\, u_{s_3}(\vec{p}_3)\, \bar{u}_{s_2}(\vec{p}_2)\, u_{s_4}(\vec{p}_4)\Big) = \frac{-g^4}{4(t-M^2)(u-M^2)}\,\cdot$

$\cdot \sum_{spins}\Big(\bar{u}_{s_1}(\vec{p}_1)\, u_{s_4}(\vec{p}_4)\, \bar{u}_{s_4}(\vec{p}_4)\, u_{s_2}(\vec{p}_2)\, \bar{u}_{s_2}(\vec{p}_2)\, u_{s_3}(\vec{p}_3)\, \bar{u}_{s_3}(\vec{p}_3)\, u_{s_1}(\vec{p}_1) +$

$+ \bar{u}_{s_2}(\vec{p}_2)\, u_{s_4}(\vec{p}_4)\, \bar{u}_{s_4}(\vec{p}_4)\, u_{s_1}(\vec{p}_1)\, \bar{u}_{s_1}(\vec{p}_1)\, u_{s_3}(\vec{p}_3)\, \bar{u}_{s_3}(\vec{p}_3)\, u_{s_2}(\vec{p}_2)\Big) = \frac{-g^4}{4(t-M^2)(u-M^2)}\,\cdot$

$\cdot\Big(tr\left[(\not{p}_1+m)(\not{p}_4+m)(\not{p}_2+m)(\not{p}_3+m)\right] + tr\left[(\not{p}_2+m)(\not{p}_4+m)(\not{p}_1+m)(\not{p}_3+m)\right]\Big) =$

$tr[\gamma^\mu\gamma^\nu\gamma^\rho] = 0 \qquad tr[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] = 4(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho})$

$= \frac{-2g^4}{(t-M^2)(u-M^2)}\Big((p_1\cdot p_4)(p_2\cdot p_3) - (p_1\cdot p_2)(p_4\cdot p_3) + (p_1\cdot p_3)(p_2\cdot p_4) +$

$+ m^2(p_1\cdot p_4 + p_1\cdot p_2 + p_1\cdot p_3 + p_2\cdot p_3 + p_2\cdot p_4 + p_3\cdot p_4) + m^4\Big)$

Use: $p_1\cdot p_2 = p_3\cdot p_4 = \frac{s}{2} - m^2$

$p_1\cdot p_3 = p_2\cdot p_4 = m^2 - \frac{t}{2}$

$p_1\cdot p_4 = p_2\cdot p_3 = m^2 - \frac{u}{2}$

$s + t + u = 4m^2$

$s = (p_1+p_2)^2 = (p_3+p_4)^2$

$t = (p_1-p_3)^2 = (p_4-p_2)^2$

$u = (p_1-p_4)^2 = (p_3-p_2)^2$

$s+t+u = 4m^2$

$p_i^2 = m^2$; $i = 1,2,3,4$

$$① = \frac{4g^4}{(t-M^2)^2}\left(2m^2-\frac{t}{2}\right)^2 = g^4\frac{(t-4m^2)^2}{(t-M^2)^2}$$

$$② = g^4\frac{(u-4m^2)^2}{(u-M^2)^2}$$

$$③ = \frac{-2g^4}{(t-M^2)(u-M^2)}\Bigg(\left(m^2-\frac{u}{2}\right)^2-\left(m^2-\frac{s}{2}\right)^2+\left(m^2-\frac{t}{2}\right)^2+$$

$$+2m^2\left[\left(m^2-\frac{u}{2}\right)-\left(m^2-\frac{s}{2}\right)+\left(m^2-\frac{t}{2}\right)\right]+m^4-m^4+m^4\Bigg)=$$

$$2m^2=\frac{s}{2}+\frac{t}{2}+\frac{u}{2}$$

$$= \frac{-2g^4}{(t-M^2)(u-M^2)}\left(\left(2m^2-\frac{u}{2}\right)^2-\left(2m^2-\frac{s}{2}\right)^2+\left(2m^2-\frac{t}{2}\right)^2\right)=$$

$$= \frac{-2g^4}{(t-M^2)(u-M^2)}\frac{1}{4}\left(s^2+2st+\cancel{t^2}-\cancel{t^2}-2ut-\cancel{u^2}+s^2+2su+\cancel{u^2}\right)=$$

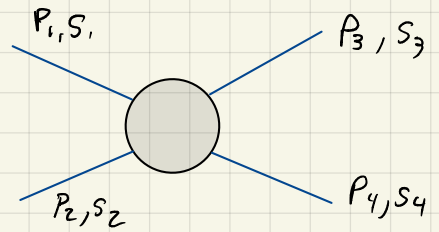
$$= \frac{g^4}{(t-M^2)(u-M^2)}\left(-s\overbrace{[s+t+u]}^{=4m^2}+ut\right) = g^4\frac{ut-4sm^2}{(t-M^2)(u-M^2)}$$

$$\Rightarrow |\overline{\mathcal{M}}|^2 = g^4\left[\frac{(t-4m^2)^2}{(t-M^2)^2}+\frac{(u-4m^2)^2}{(u-M^2)^2}+\frac{ut-4sm^2}{(t-M^2)(u-M^2)}\right]$$

(5)

$$\frac{d\sigma}{d\Omega} = \frac{|\overline{\mathcal{M}}|^2}{64\pi^2E^2} = \frac{|\overline{\mathcal{M}}|^2}{64\pi^2 s}$$

C)



$\psi\psi \to \psi\psi$

$$|i\rangle = a^+_{s_1}(\vec{p}_1)\, a^+_{s_1}(\vec{p}_2)|0\rangle; \quad |f\rangle = a^+_{s_3}(\vec{p}_3)\, a^+_{s_4}(\vec{p}_4)|0\rangle$$

(1) $\langle f|(-i)\int_{-\infty}^{\infty} dt\, T H_{int}|i\rangle = -ig\int d^4x \langle f|T\{\bar{\Psi}(x)\gamma_5\Psi(x)\varphi(x)\}|i\rangle = 0$

by Wick's Theorem: only fully contracted set of fields can give rise to a non-zero vaccum expectation value. An odd number of field can not be fully contracted as eak Wick contraction involves a pair of fields.

$$[a^{(+)}, c^{(+)}] = [b^{(+)}, c^{(+)}] = 0 \quad \& \quad c|0\rangle = 0$$

Reminder: $T\{\varphi(x_1)\ldots\varphi(x_n)\} = N\{\varphi(x_1)\ldots\varphi(x_n) + \text{all p.c.}\}$

also: $\langle 0|N(\text{any operator})|0\rangle = 0$

(2) $T\{\bar{\Psi}(x)\gamma_5\Psi(x)\varphi(x)\ \bar{\Psi}(x')\gamma_5\Psi(x')\varphi(x')\}$

$$= :\bar{\Psi}(x)\gamma_5\Psi(x)\ \bar{\Psi}(x')\gamma_5\Psi(x'): \ \underbrace{\varphi(x)\,\varphi(x')}_{D_F(x,x')}$$

$\int_{x,x'} = \int d^4x \int d^4x$

(3) $\Rightarrow I \equiv \int_{x,x'} \langle f|:\bar{\Psi}(x)\gamma_5\Psi(x)\bar{\Psi}(x')\gamma_5\Psi(x'):|i\rangle D_F(x,x')$

$$\Psi(x) = \int_p \sum_i [u_i(\vec{p})a_i(\vec{p})e^{-ip\cdot x} + v_i(\vec{p})b_i^+(\vec{p})e^{ip\cdot x}; \quad \int_p \equiv \frac{d^3\vec{p}}{(2\pi)^3 2\omega_p}$$

$$\hookrightarrow I = \int_{x,x',p,q,p',q'} \sum_{i,j,k,l} \langle f| : \bar{u}_i(\vec{p}) a_i^+(\vec{p}) e^{ipx} \gamma_5 u_j(\vec{q}) a_j(\vec{q}) e^{-iqx} \times$$
$$\times \bar{u}_K(\vec{p}\,') a_K^+(\vec{p}\,') e^{ip'x'} \gamma_5 u_l(\vec{q}\,') a_l(\vec{q}\,') e^{-iq'x'} : |i\rangle$$
$$\times \mathcal{D}_F(x,x')$$

$$\cancel{f}$$

$$= -\int_{\substack{x,x',p,q,p',q' \\ i,j,k,l}} \sum \bar{u}_i(\vec{p}) \gamma_5 u_j(\vec{q}) \bar{u}_K(\vec{p}\,') \gamma_5 u_l(\vec{q}\,') e^{ipx} e^{-iqx} e^{ip'x'} e^{-iq'x'} \times \mathcal{D}_F(x,x')$$

$$\langle 0|\cdots|0\rangle \leftarrow \times \langle 0| a_{s_4}(\vec{p}_4) \underbrace{a_{s_3}(p_3) a_i^+(\vec{p}) a_K^+}_{} (\vec{p}\,') a_j(\vec{q}) a_l(q') a_{s_1}^+(\vec{p}_1) a_{s_2}^+(\vec{p}_2)|0\rangle$$

$$= -a_i^+(\vec{p}) a_{s_3}(\vec{p}_3) + (2\pi)^3 2\omega_{\vec{p}} \delta_{i,s_3} \delta^{(3)}(\vec{p}-\vec{p}_3)$$
$$= -a_K^+(\vec{p}\,') a_{s_3}(\vec{p}_3) + (2\pi)^3 2\omega_{\vec{p}'} \delta_{K,s_3} \delta^{(3)}(\vec{p}\,'-\vec{p}_3)$$

$$\langle 0|\cdots|0\rangle = \langle 0| \Big[-\underbrace{a_{s_4}(\vec{p}_4) a_i^+(\vec{p})}_{= -a_i^+(\vec{p}) a_{s_4}(\vec{p}_4) + (2\pi)^3 2\omega_{\vec{p}} \delta_{i,s_4} \delta^{(3)}(\vec{p}-\vec{p}_4)} a_{s_3}(\vec{p}_3) a_K^+(\vec{p}\,') + (2\pi)^3 2\omega_{\vec{p}} \delta_{i,s_3} \delta^{(3)}(\vec{p}-\vec{p}_3) \underbrace{a_{s_4}(\vec{p}_4) \times a_K^+(\vec{p}\,')}_{= -a_K^+(\vec{p}\,') a_4(p_4) + (2\pi)^3 2\omega_{\vec{p}'} \delta_{K,s_4} \delta^{(3)}(\vec{p}\,'-\vec{p}_4)} \Big]$$
$$\times a_j(\vec{q}) a_l(\vec{q}) a_{s_1}^+(\vec{p}_1) a_{s_2}^+(\vec{p}_2)|0\rangle$$

$$= \Big[-(2\pi)^3 2\omega_{\vec{p}} \delta_{i,s_4} \delta^{(3)}(\vec{p}-\vec{p}_4) (2\pi)^3 2\omega_{\vec{p}'} \delta_{K s_3} \delta^{(3)}(\vec{p}\,'-\vec{p}_3)$$
$$+ (2\pi)^3 2\omega_{\vec{p}} \delta_{i,s_3} \delta^{(3)}(\vec{p}-\vec{p}_3)(2\pi)^3 2\omega_{\vec{p}'} \delta_{K s_4} \delta^{(3)}(\vec{p}\,'-p_4) \Big] \cdot$$
$$\cdot \Big[-(2\pi)^3 2\omega_{\vec{q}} \delta_{j,s_4} \delta^{(3)}(\vec{q}-\vec{p}_1)(2\pi)^3 2\omega_{\vec{q}'} \delta_{l,s_2} \delta^{(3)}(\vec{q}\,'-\vec{p}_2) +$$
$$+ (2\pi)^3 2\omega_{\vec{q}} \delta_{j,s_2} \delta^{(3)}(\vec{q}-p_2)(2\pi)^3 2\omega_{\vec{q}'} \delta_{l s_1} \delta^{(3)}(q'-p_1) \Big]$$

$$\hookrightarrow I = \int_{x,x',q,q'} \sum_{j,l} \Big(\bar{u}_{s_4}(\vec{p}_4) e^{ip_4 x} \gamma_5 u_j(\vec{q}) e^{-iqx} \bar{u}_{s_3}(\vec{p}_3) e^{ip_3 x'} \gamma_5 u_l(\vec{q}\,') e^{-iq'x'}$$
$$- \bar{u}_{s_3}(\vec{p}_3) e^{ip_3 x} \gamma_5 u_j(\vec{q}) e^{-iqx} \bar{u}_{s_4}(\vec{p}_4) e^{ip_4 x'} \gamma_5 u_l(\vec{q}\,') e^{-iq'x'} \Big)$$

$$\times \Big[(2\pi)^3 2\omega_{\vec{q}} \delta_{j,s_2} \delta^{(3)}(\vec{q}-\vec{p}_2)(2\pi)^3 2\omega_{\vec{q}'} \delta_{l,s_1} \delta^{(3)}(\vec{q}\,'-\vec{p}_1)$$
$$- (2\pi)^3 2\omega_{\vec{q}} \delta_{j,s_1} \delta^{(3)}(\vec{q}_1-\vec{p}_1)(2\pi)^3 2\omega_{\vec{q}'} \delta_{l,s_2} \delta^{(3)}(\vec{q}\,'-p_2) \Big]$$
$$\times \mathcal{D}_F(x,x')$$

$$\boxed{u_i \equiv u_{s_i}(\vec{p}_i)}$$

$$\hookrightarrow I = \int_{x,x'} \Big(\bar{u}_4 \gamma_5 u_2 \, \bar{u}_3 \gamma_5 u_1 \, e^{i(p_4 - p_2)x} e^{i(p_3 - p_1)x'}$$
$$- \bar{u}_4 \gamma_5 u_1 \bar{u}_3 \gamma_5 u_2 \, e^{i(p_4 - p_1)x} e^{i(p_3 - p_2)x'}$$
$$- \bar{u}_3 \gamma_5 u_2 \, \bar{u}_4 \gamma_5 u_1 \, e^{i(p_3 - p_2)x} e^{i(p_4 - p_1)x'}$$
$$+ \bar{u}_3 \gamma_5 u_1 \, \bar{u}_4 \gamma_5 u_2 \, e^{i(p_3 - p_1)x} e^{i(p_4 - p_2)x'} \, \underbrace{D_F(x,x')}_{= \int \frac{d^4p}{(2\pi)^4} \frac{i e^{-ip(x-x')}}{p^2 - M^2}}$$

$$\hookrightarrow I = 2(2\pi)^4 \int \frac{d^4p}{p^2 - M^2} \, i \Big(\delta^{(4)}(p_4 - p_2 - p)\delta^{(4)}(p_3 - p_1 - p) \, \bar{u}_4 \gamma_5 u_2 \, \bar{u}_3 \gamma_5 u_1$$
$$- \delta^{(4)}(p_4 - p_2 - p) \, \delta^{(4)}(p_3 - p_2 + p) \bar{u}_4 \gamma_5 u_1 \bar{u}_3 \gamma_5 u_2 \Big)$$

$$= (2\pi)^4 \, \delta^{(4)}(p_3 + p_4 - p_1 - p_2) \, i \, 2 \left(\frac{\bar{u}_4 \gamma_5 u_2 \, \bar{u}_3 \gamma_5 u_1}{(p_1 - p_3)^2 - M^2} - \frac{\bar{u}_4 \gamma_5 u_1 \bar{u}_3 \gamma_5 u_2}{(p_1 - p_4)^2 - M^2} \right)$$

$$\hookrightarrow \mathcal{M} = -g^2 \left(\frac{\bar{u}_4 \gamma_5 u_2 \, \bar{u}_3 \gamma_5 u_1}{t - M^2} - \frac{\bar{u}_4 \gamma_5 u_1 \, \bar{u}_3 \gamma_5 u_2}{u - M^2} \right)$$
$$\equiv \mathcal{M}_t + \mathcal{M}_u$$

4) $$|\overline{\mathcal{M}}|^2 = \frac{1}{4} \sum_{\text{spins}} |\mathcal{M}|^2 = \frac{1}{4} \sum_{\text{spins}} \underbrace{|\mathcal{M}_t|^2}_{1} + \underbrace{|\mathcal{M}_u|^2}_{2} + \underbrace{\mathcal{M}_t \mathcal{M}_u^+ + \mathcal{M}_u \mathcal{M}_t^+}_{3}$$

[1 $$\frac{1}{4} \sum |\mathcal{M}_t|^2 \underset{\uparrow \, \gamma_5^+ = \gamma_5}{=} \frac{g^4}{4(t - M^2)^2} \sum \bar{u}_4 \gamma_5 u_2 \, \bar{u}_3 \gamma_5 \underbrace{u_1 \bar{u}_1} \gamma_5 u_3 \, \bar{u}_2 \gamma_5 u_4$$

Recall:

$$\sum_{s_1} u_{s_1}(\vec{p}_1) \bar{u}_{s_1}(\vec{p}_1) = (\not{p}_1 + m)$$

$$= \frac{g^4}{4(t - M^2)^2} \, tr[\gamma_5 (\not{p}_1 + m) \gamma_5 (\not{p}_3 + m)] \, tr[\gamma_5 (\not{p}_2 + m) \gamma_5 (\not{p}_4 + m)]$$

$$\underset{\uparrow \, \{\gamma_5, \gamma^\mu\} = 0}{=} \frac{4g^4}{(t - M^2)^2} \Big(\underbrace{-p_1 p_3}_{= -p_2 \cdot p_4 = \frac{t}{2} - m^2} + m^2 \Big) \Big(-p_2 p_4 + m^2 \Big)$$

$$= \boxed{\frac{g^4 t^2}{(t - M^2)^2}}$$

② Similarly:

$$\frac{1}{4}\sum|\mathcal{M}_u|^2=\frac{4g^4}{(u-M^2)^2}\Big(\underbrace{-p_2\cdot p_3}_{=-p_1\cdot p_4=\frac{u}{2}-m^2}+m^2\Big)\Big(-p_1p_4+m^2\Big)=\boxed{\frac{g^4u^2}{(u-M^2)^2}}$$

③

$$\frac{1}{4}\sum \mathcal{M}_t\mathcal{M}_u^+ + \mathcal{M}_u\mathcal{M}_t^+ = \frac{-g^4}{4(t-M^2)(u-M^2)}\sum_{spins}\Big(\bar u_4\gamma_5 u_2\,\bar u_3\gamma_5 u_1\,\bar u_2\gamma_5 u_3\,\bar u_1\gamma_5 u_4{}^+$$
$$+\bar u_4\gamma_5 u_1\,\bar u_3\gamma_5 u_2\,\bar u_1\gamma_5 u_3\,\bar u_2\gamma_5 u_4\Big)$$

$$=\frac{-g^4}{4(t-M^2)(u-M^2)}\sum_{spins}\Big(\bar u_1\gamma_5 u_4\,\bar u_4\gamma_5 u_2\bar u_2\,\gamma_5 u_3\,\bar u_3\,\gamma_5 u_1 +$$
$$+\bar u_2\gamma_5 u_4\,\bar u_4\gamma_5 u_1\,\bar u_1\gamma_5 u_3\,\bar u_3\gamma_5 u_2\Big)$$

$$=\frac{-g^4}{4(t-M^2)(u-M^2)}\Big(tr\big(\gamma_5(\not p_4+m)\gamma_5(\not p_2+m)\gamma_5(\not p_3+m)\gamma_5(\not p_1+m)\big)+$$
$$+tr\big(\gamma_5(\not p_4+m)\gamma_5(\not p_1+m)\gamma_5(\not p_3+m)\gamma_5(\not p_2+m)\big)\Big)$$

$$=\frac{-g^4}{4(t-M^2)(u-M^2)}\Big(tr\big((-\not p_1+m)(\not p_4+m)(-\not p_2+m)(\not p_3+m)\big)+$$
$$+tr\big((-\not p_2+m)(\not p_4+m)(-\not p_1+m)(\not p_3+m)\big)\Big)$$

$$=\frac{-g^4}{4(t-M^2)(u-M^2)}\Big(tr(\not p_1\not p_4\not p_2\not p_3)+m^2tr(-\not p_1\not p_4+\not p_1\not p_2-\not p_1\not p_3-\not p_4\not p_2+\not p_4\not p_3-\not p_2\not p_3)$$
$$tr(\not p_2\not p_4\not p_1\not p_3)+m^2tr(-\not p_2\not p_4+\not p_2\not p_1-\not p_2\not p_3-\not p_4\not p_1+\not p_4\not p_3-\not p_1\not p_3)$$
$$+8m^4\Big)$$

$$\begin{cases} tr[\gamma^\mu\gamma^\nu]=4\eta^{\mu\nu} \\ tr[\gamma^\mu\gamma^\nu\gamma^\rho]=0 \quad \& \\ tr[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma]=4(\eta^{\mu\nu}\eta^{\rho\sigma}-\eta^{\mu\rho}\eta^{\nu\sigma}+\eta^{\mu\sigma}\eta^{\nu\rho}) \end{cases}$$

$$=\frac{-2g^4}{(t-M^2)(u-M^2)}\Big(p_1\cdot p_4\,p_2\cdot p_3-p_1\cdot p_2\,p_4\cdot p_3+p_1\cdot p_3\,p_4\cdot p_2$$
$$+m^2(-p_1\cdot p_4+p_1p_2-p_1\cdot p_3-p_4\cdot p_2+p_4\cdot p_3-p_2\cdot p_3)+m^4\Big)$$

$$=\frac{-2g^4}{(t-M^2)(u-M^2)}\Big(\big(m^2-\tfrac{u}{2}\big)^2-\big(\tfrac{s}{2}-m^2\big)^2+\big(m^2-\tfrac{t}{2}\big)^2+2m^2\Big(-\big(m^2-\tfrac{u}{2}\big)+\big(\tfrac{s}{2}-m^2\big)-\big(m^2-\tfrac{t}{2}\big)\Big)+m^4\Big)$$

$$= \frac{-2g^4}{(t-M^2)(u-M^2)}\left(2m^4+m^2(s-u-t)-\frac{s^2}{4}+\frac{u^2}{4}+\frac{t^2}{4}+\underbrace{2m^2\left(-3m^2+\frac{s}{2}+\frac{u}{2}+\frac{t}{2}\right)}_{\substack{=2m^2(-3m^2+2m^2)\\ =-2m^4}}\right)$$

$$= \frac{-2g^4}{(t-M^2)(u-M^2)}\left(m^2(s-u-t)-\frac{1}{4}\underbrace{(s^2-u^2-t^2)}_{\substack{=(s-u-t)(s+u+t)+2ut\\ =4m^2(s-u-t)+2ut}}\right)$$

$$= \frac{-2g^4}{(t-M^2)(u-M^2)}\left(-\frac{1}{2}ut\right) = \boxed{\frac{g^4 ut}{(t-M^2)(u-M^2)}}$$

$$\boxed{\Rightarrow |\bar{\mathcal{M}}|^2 = g^4\left(\frac{t^2}{(t-M^2)^2}+\frac{u^2}{(u-M^2)^2}+\frac{ut}{(t-M^2)(u-M^2)}\right)}$$

5) $\frac{d\sigma}{d\Omega} = \frac{|\bar{\mathcal{M}}|^2}{64\pi^2 s}$

Maxwell theory

PART A:

① Starting from

$$\partial^\mu F_{\mu\nu} = -j_\nu ,$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, we find

$$\partial^\mu(\partial_\mu A_\nu - \partial_\nu A_\mu) = -j_\nu$$

$$\rightarrow \partial^2 A_\nu - \partial_\nu \partial^\mu A_\mu = -j_\nu$$

$$\rightarrow (\eta_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)A^\nu = -j_\mu$$

Using $f(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ikx} f(k)$,

we obtain

$$P_{\mu\nu} A^\nu(k) = -j_\mu(k) , \quad \text{with}$$

$$P_{\mu\nu} = -k^2\eta_{\mu\nu} + k_\mu k_\nu .$$

② $P_{\mu\nu}$ is the following 4x4 matrix

$$P_{\mu\nu} = \begin{pmatrix} -k^2+(k_0)^2 & k_0k_1 & k_0k_2 & k_0k_3 \\ k_0k_1 & k^2+(k_1)^2 & k_1k_2 & k_1k_3 \\ k_0k_2 & k_1k_2 & k^2+(k_2)^2 & k_2k_3 \\ k_0k_3 & k_1k_3 & k_2k_3 & k^2+(k_3)^2 \end{pmatrix} .$$

Notice that $\det(P_{\mu\nu}) = 0 \rightarrow$ degenerate matrix, due to the gauge invariance.

(3) In the Lorentz gauge, $k_\mu A^\mu = 0$, the Maxwell equations boil down to

$$k^2 A_\mu(k) = J_\mu(k) \rightsquigarrow A_\mu(k) = \frac{J_\mu(k)}{k^2}.$$

– In the Coulomb gauge, $k_i A^i = 0$, we notice that the Maxwell equations become:

$$k^2 A_\mu(k) - k_\mu k_0 A_0(k) = J_\mu(k)$$

For $\mu = 0$, we find

$$\boxed{A_0(k) = -\frac{J_0(k)}{\vec{k}^2}} \quad \circledast$$

For $\mu = i$, we find

$$k^2 A_i(k) - k_i k_0 A_0(k) = J_i(k)$$

$$\overset{\circledast}{\rightsquigarrow} A_i(k) = \frac{1}{k^2}\left(J_i(k) - \frac{k_i k_0 J_0(k)}{\vec{k}^2}\right)$$

Since $k_\mu J^\mu = 0 \rightarrow k_0 J_0(k) = k_j J_j(k)$, we find:

$$\boxed{A_i(k) = \frac{1}{k^2}\left(\delta_{ij} - \frac{k_i k_j}{\vec{k}^2}\right) J_j(k)}$$

PART B:

① $$\partial_\mu A^\mu = i\int \frac{d^3\vec{k}}{(2\pi)^3 2w_{\vec{k}}} \sum_{r=0}^{3} k_\mu \left(-\varepsilon_r^\mu(\vec{k}) a_r(\vec{k}) e^{-ikx} + \varepsilon_r^\mu(\vec{k}) a_r^*(\vec{k}) e^{ikx}\right)$$

Since $k_\mu \varepsilon_1^\mu = k_\mu \varepsilon_2^\mu = 0$, while $k_\mu \varepsilon_0^\mu = -k_\mu \varepsilon_3^\mu = w_{\vec{k}}$, we find:

$$\rightsquigarrow \partial_\mu A^\mu = \frac{i}{2}\int \frac{d^3\vec{k}}{(2\pi)^3}\left[-\left(a_0(\vec{k}) - a_3(\vec{k})\right)e^{-ikx} + \left(a_0^*(\vec{k}) - a_3^*(\vec{k})\right)e^{ikx}\right],$$

$\rightarrow a_0(\vec{k}) = a_3(\vec{k})$ $\left(\& \; a_0^*(\vec{k}) = a_3^*(\vec{k})\right)$, in order for the above to vanish.

② Let's show that $\eta_{\mu\nu}\varepsilon_r^\mu \varepsilon_s^\nu = -\zeta_s \delta_{rs}$,

working case by case:

⊛ <u>$r=0, s=0$</u>

$\eta_{\mu\nu}\varepsilon_0^\mu\varepsilon_0^\nu = 1 = -\zeta_0\delta_{00}$, with $\zeta_0 = -1$

⊛ <u>$r=0, s\neq 0$</u>

$\eta_{\mu\nu}\varepsilon_0^\mu\varepsilon_s^\nu = 0$

⊛ <u>$r=1, s=1$</u>

$\eta_{\mu\nu}\varepsilon_1^\mu\varepsilon_1^\nu = -1 = -\zeta_1\delta_{11}$, with $\zeta_1 = 1$

⊛ <u>$r=1, s\neq 1$</u>

$\eta_{\mu\nu}\varepsilon_1^\mu\varepsilon_s^\nu = 0$

* $r=2,\ s=2$

$$\eta_{\mu\nu}\varepsilon_2^\mu\varepsilon_2^\nu = -1 = -\zeta_2\delta_{22}, \text{ with } \zeta_2 = 1$$

* $r=2,\ s\neq 2$

$$\eta_{\mu\nu}\varepsilon_2^\mu\varepsilon_s^\nu = 0$$

* $r=3,\ s=3$

$$\eta_{\mu\nu}\varepsilon_3^\mu\varepsilon_3^\nu = -1 = -\zeta_3\delta_{33}, \text{ with } \zeta_3 = 1$$

* $r=3,\ s\neq 3$

$$\eta_{\mu\nu}\varepsilon_3^\mu\varepsilon_s^\nu = 0$$

Putting everything together, we arrive at

$$\underline{\eta_{\mu\nu}\varepsilon_r^\mu\varepsilon_s^\nu = -\zeta_s\delta_{rs}}$$

$$\sum_{r=0}^{3}\zeta_r\varepsilon_r^\mu\varepsilon_r^\nu = -\varepsilon_0^\mu\varepsilon_0^\nu + \varepsilon_1^\mu\varepsilon_1^\nu + \varepsilon_2^\mu\varepsilon_2^\nu + \varepsilon_3^\mu\varepsilon_3^\nu$$

$$= -\begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix} + \begin{pmatrix}0&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix} + \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix} + \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1\end{pmatrix}$$

$$= -\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix} = -\eta^{\mu\nu}$$

3, 4 The action of the Maxwell theory reads

$$S = -\frac{1}{4}\int d^4x\, F_{\mu\nu}F^{\mu\nu}$$

The canonical energy-momentum tensor is given by:

$$T^{\mu\nu} = \frac{\delta \mathcal{L}}{\delta \partial_\mu A_\rho} \partial^\nu A_\rho - \eta^{\mu\nu} \mathcal{L}$$

$$\rightarrow T^{\mu\nu} = - F^{\mu\rho} \partial^\nu A_\rho + \frac{1}{4} \eta^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}$$

Using $A^\mu = \int \widetilde{dk} \sum_{r=0}^{3} \varepsilon_r^\mu (a_r(\vec{k}) e^{-ikx} + a_r^*(\vec{k}) e^{ikx})$, with $\widetilde{dk} = \frac{d^3\vec{k}}{(2\pi)^3 2\omega_k}$,

we will express $T^{\mu\nu}$ in terms of a's. First we compute:

$$\partial_\alpha A_\beta = -i \int \widetilde{dk} \sum_{r=0}^{3} k_\alpha \varepsilon_{\beta,r} (a_r(\vec{k}) e^{-ikx} - a_r^*(\vec{k}) e^{ikx})$$

$$\rightarrow F_{\alpha\beta} = -i \int \widetilde{dk} \sum_{r=0}^{3} (k_\alpha \varepsilon_{\beta,r} - k_\beta \varepsilon_{\alpha,r}) (a_r(\vec{k}) e^{-ikx} - a_r^*(\vec{k}) e^{ikx})$$

Therefore,

$$-F^{\mu\rho}\partial^\nu A_\rho = \int \widetilde{dk}\, \widetilde{dk}' \sum_{r,s=0}^{3} (k^\mu k'^\nu \varepsilon_r^\rho \varepsilon_{\rho,s} - k^\rho \varepsilon_{\rho,s} k'^\nu \varepsilon_r^\mu) (a_r(\vec{k}) e^{-ikx} - a_r^*(\vec{k}) e^{ikx}) \cdot (a_s(\vec{k}') e^{-ik'x} - a_s^*(\vec{k}') e^{ik'x})$$

Using $\varepsilon_r^\rho \varepsilon_{\rho,s} = -\zeta_s \delta_{rs}$, the above becomes:

$$-F^{\mu\rho}\partial^\nu A_\rho = -\int \widetilde{dk}\, \widetilde{dk}'\, k^\mu k'^\nu \Big\{ \sum_{r=0}^{3} \zeta_r (a_r(\vec{k}) e^{-ikx} - a_r^*(\vec{k}) e^{ikx}) \cdot (a_r(\vec{k}') e^{-ik'x} - a_r^*(\vec{k}') e^{ik'x}) + \sum_{r,s=0}^{3} k^\rho \varepsilon_{\rho,s} k'^\nu \varepsilon_r^\mu (a_r(\vec{k}) e^{-ikx} - a_r^*(\vec{k}) e^{ikx}) \cdot (a_s(\vec{k}') e^{-ik'x} - a_s^*(\vec{k}') e^{ik'x}) \Big\}$$

Now we look at

$$F_{\alpha\beta}F^{\alpha\beta} = -\int \widetilde{dk}\,\widetilde{dk}' \sum_{r,s=0}^{3} (k_\alpha \varepsilon_{\beta,r} - k_\beta \varepsilon_{\alpha,r})(k'^\alpha \varepsilon_s^\beta - k'^\beta \varepsilon_s^\alpha)(a_r(\vec{k})e^{-ikx} - a_r^*(\vec{k})e^{ikx}) \cdot$$
$$\cdot (a_s(\vec{k}')e^{-ik'x} - a_s^*(\vec{k}')e^{ik'x})$$

$$= -2\int \widetilde{dk}\,\widetilde{dk}' \sum_{r,s=0}^{3} (k_\alpha k'^\alpha \varepsilon_{\beta,r}\varepsilon_s^\beta - k_\alpha \varepsilon_s^\alpha k'^\beta \varepsilon_{\beta,r})(a_r(\vec{k})e^{-ikx} - a_r^*(\vec{k})e^{ikx}) \cdot$$
$$\cdot (a_s(\vec{k}')e^{-ik'x} - a_s^*(\vec{k}')e^{ik'x})$$

$$= 2\int \widetilde{dk}\,\widetilde{dk}' \Big\{ k_\alpha k'^\alpha \sum_{r=0}^{3} \zeta_r (a_r(\vec{k})e^{-ikx} - a_r^*(\vec{k})e^{ikx})(a_r(\vec{k}')e^{-ik'x} - a_r^*(\vec{k}')e^{ik'x})$$
$$+ \sum_{r,s=0}^{3} k_\alpha \varepsilon_s^\alpha k'^\beta \varepsilon_{\beta,r} (a_r(\vec{k})e^{-ikx} - a_r^*(\vec{k})e^{ikx})(a_s(\vec{k}')e^{-ik'x} - a_s^*(\vec{k}')e^{ik'x}) \Big\}$$

Putting together the two pieces above, we obtain

$$T^{\mu\nu} = \int \widetilde{dk}\,\widetilde{dk}' \Big\{ (-k^\mu k'^\nu + \frac{1}{2}\eta^{\mu\nu} k_\alpha k'^\alpha) \sum_{r=0}^{3} \zeta_r (a_r(\vec{k})e^{-ikx} - a_r^*(\vec{k})e^{ikx})(a_r(\vec{k}')e^{-ik'x} - a_r^*(\vec{k}')e^{ik'x})$$
$$+ \sum_{r,s=0}^{3} k^\beta \varepsilon_{\beta,s} (-k'^\nu \varepsilon_r^\mu + \frac{1}{2}\eta^{\mu\nu} k'^\alpha \varepsilon_{\alpha,r})(a_r(\vec{k})e^{-ikx} - a_r^*(\vec{k})e^{ikx})(a_s(\vec{k}')e^{-ik'x} - a_s^*(\vec{k}')e^{ik'x}) \Big\}$$

The Hamiltonian can be immediately found from the above as

$$H = \int d^3\vec{x}\, T^{00} = \int d^3\vec{x}\, \widetilde{dk}\,\widetilde{dk}' \Big\{ (-k^0 k'^0 + \frac{1}{2} k_\alpha k'^\alpha) \sum_{r=0}^{3} \zeta_r (a_r(\vec{k})e^{-ikx} - a_r^*(\vec{k})e^{ikx}) \cdot$$
$$\cdot (a_r(\vec{k}')e^{-ik'x} - a_r^*(\vec{k}')e^{ik'x}) + \sum_{r,s=0}^{3} k^\beta \varepsilon_{\beta,s} (-k'^0 \varepsilon_r^0 + \frac{1}{2} k'^\alpha \varepsilon_{\alpha,r})(a_r(\vec{k})e^{-ikx} - a_r^*(\vec{k})e^{ikx}) \cdot$$
$$\cdot (a_s(\vec{k}')e^{-ik'x} - a_s^*(\vec{k}')e^{ik'x}) \Big\} = H_1 + H_2$$

Let's now carry out the integration over x in H_1.

$$H_1 = \int d\tilde{k}\, \frac{d^3k'}{2\omega_{\vec{k}'}} \Big\{ \left(-k^0 k'^0 + \frac{1}{2} k_\alpha k'^\alpha\right) \sum_{r=0}^{3} \zeta_r \Big[a_r(\vec{k}) a_r(\vec{k}') e^{-i(\omega_{\vec{k}}+\omega_{\vec{k}'})t} \delta^{(3)}(\vec{k}+\vec{k}')$$

$$- a_r(\vec{k}) a_r^*(\vec{k}') e^{-i(\omega_{\vec{k}}-\omega_{\vec{k}'})t} \delta^{(3)}(\vec{k}-\vec{k}') - a_r^*(\vec{k}) a_r(\vec{k}') e^{i(\omega_{\vec{k}}-\omega_{\vec{k}'})t} \delta^{(3)}(\vec{k}-\vec{k}')$$

$$+ a_r^*(\vec{k}) a_r^*(\vec{k}') e^{i(\omega_{\vec{k}}+\omega_{\vec{k}'})t} \delta^{(3)}(\vec{k}+\vec{k}') \Big]$$

Performing the integration over k', we find

$$H_1 = \frac{1}{2} \int \frac{d^3\vec{k}}{(2\pi)^3 2\omega_{\vec{k}}^2} \sum_{r=0}^{3} \zeta_r \Big[\Big(-\omega_{\vec{k}}^2 + \frac{1}{2}(\omega_{\vec{k}}^2 + \vec{k}^2)\Big) a_r(\vec{k}) a_r(-\vec{k}) e^{-2i\omega_k t}$$

$$-2\Big(-\omega_{\vec{k}}^2 + \frac{1}{2}(\omega_{\vec{k}}^2 - \vec{k}^2)\Big) a_r(\vec{k}) a_r^*(\vec{k}) + \Big(-\omega_{\vec{k}}^2 + \frac{1}{2}(\omega_{\vec{k}}^2 + \vec{k}^2)\Big) a_r^*(\vec{k}) a_r^*(-\vec{k}) e^{2i\omega_k t} \Big]$$

Using the fact that $k^2 = \omega_{\vec{k}}^2 - \vec{k}^2 = 0$, the above boils down to

$$H_1 = \int \frac{d^3\vec{k}}{2(2\pi)^3} \sum_{r=0}^{3} \zeta_r\, a_r(\vec{k}) a_r^*(\vec{k})$$

Since $a_0(\vec{k}) = a_3(\vec{k})$ & $a_0^*(\vec{k}) = a_3^*(\vec{k})$, we notice that

$$H_1 = \int \frac{d^3\vec{k}}{2(2\pi)^3} \sum_{r=1,2} a_r(\vec{k}) a_r^*(\vec{k}) . \qquad (*)$$

Similarly for H_2, we find

$$H_2 = \frac{1}{2} \int \frac{d^3\vec{k}}{(2\pi)^3 2\omega_{\vec{k}}^2} \Big\{ \sum_{r,s=0}^{3} k^\rho \varepsilon_{\rho,s} \Big[\Big(-\frac{1}{2} k^0 \varepsilon_r^0 + \frac{1}{2} \vec{k}\cdot\vec{\varepsilon}_r\Big) a_r(\vec{k}) a_s(-\vec{k}) e^{-2i\omega_{\vec{k}} t}$$

$$-\Big(-\frac{1}{2} k^0 \varepsilon_r^0 - \frac{1}{2} \vec{k}\cdot\vec{\varepsilon}_r\Big) a_r(\vec{k}) a_s^*(\vec{k}) - \Big(-\frac{1}{2} k^0 \varepsilon_r^0 - \frac{1}{2} \vec{k}\cdot\vec{\varepsilon}_r\Big) a_r^*(\vec{k}) a_s(\vec{k})$$

$$+ \Big(-\frac{1}{2} k^0 \varepsilon_r^0 + \frac{1}{2} \vec{k}\cdot\vec{\varepsilon}_r\Big) a_r^*(\vec{k}) a_s^*(-\vec{k})\, e^{2i\omega_{\vec{k}} t} \Big] \Big\}$$

At this stage, we note that $k^\mu = \omega_{\vec{k}}(1,0,0,1)$,

$\varepsilon_0^\mu = (1,0,0,0)$, $\varepsilon_1^\mu = (0,1,0,0)$, $\varepsilon_2^\mu = (0,0,1,0)$, $\varepsilon_3^\mu = (0,0,0,1)$.

Then we find that $H_2 = 0$, since:

$$\sum_{s=0}^{3} k^\rho \varepsilon_{\rho,s}\, a_s(-\vec{k}) \sum_{r=0}^{3} \left(-\frac{1}{2} k^0 \varepsilon_r^0 + \frac{1}{2}\vec{k}\cdot\vec{\varepsilon}_r\right) a_r(\vec{k})\, e^{-2i\omega_{\vec{k}} t}$$

$$= \omega_{\vec{k}}\left(a_0(-\vec{k}) - a_3(-\vec{k})\right) \sum_{r=0}^{3} \left(-\frac{1}{2} k^0 \varepsilon_r^0 + \frac{1}{2}\vec{k}\cdot\vec{\varepsilon}_r\right) a_r(\vec{k})\, e^{-2i\omega_{\vec{k}} t}\Big|_{a_0=a_3} = 0$$

$$\sum_{s=0}^{3} k^\rho \varepsilon_{\rho,s}\, a_s^*(\vec{k}) \sum_{r=0}^{3} \left(-\frac{1}{2} k^0 \varepsilon_r^0 - \frac{1}{2}\vec{k}\cdot\vec{\varepsilon}_r\right) a_r(\vec{k})$$

$$= \omega_{\vec{k}}\left(a_0^*(\vec{k}) - a_3^*(\vec{k})\right) \sum_{r=0}^{3} \left(-\frac{1}{2} k^0 \varepsilon_r^0 - \frac{1}{2}\vec{k}\cdot\vec{\varepsilon}_r\right) a_r(\vec{k})\Big|_{a_0=a_3} = 0$$

e.t.c .

Therefore the Hamiltonian is simply

$$H = \int \frac{d^3\vec{k}}{2(2\pi)^3} \sum_{r=1,2} a_r(\vec{k})\, a_r^*(\vec{k}).$$