Quantum Field Theory (Quantum Electrodynamics)

Problem Set 11 15 & 17 January 2024

1. Fermion Scattering in Yukawa theory

Let us consider the following Lagrangian

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi + \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}M^2\phi^2 - g\phi\bar{\psi}\psi$$

where ϕ is a real scalar field, $g > 0$ and $m,\ M$ are the fermionic and scalar masses, respectively.

Part A

Let us study the scattering process $\psi\psi \to \psi\psi$ in the context of the Yukawa theory. This means that the initial and final states are chosen as

$$|i\rangle = a^+_{s_1}(\vec{p}_1)a^+_{s_2}(\vec{p}_2)\,|0\rangle \ , \quad |f\rangle = a^+_{s_3}(\vec{p}_3)a^+_{s_4}(\vec{p}_4)\,|0\rangle \ .$$

Starting from

$$\langle f|\left(Te^{-i\int_{-\infty}^{+\infty}\mathrm{d}tH_{int}} - 1\right)|i\rangle \ ,$$

with

$$H_{int} = g\int \mathrm{d}^3\vec{x}\,\bar{\psi}(x)\psi(x)\phi(x) \ ,$$

we find that the matrix element between the initial and final states *at the lowest non-vanishing order in g* is quadratic in the coupling constant and reads

$$-\frac{g^2}{2}\int \mathrm{d}^4x\int \mathrm{d}^4x'\,\langle f|\,T\left\{\bar{\psi}(x)\psi(x)\phi(x)\bar{\psi}(x')\psi(x')\phi(x')\right\}|i\rangle \ ,$$

where T is the time-ordered product.

1. Convince yourselves that the term linear in g is zero.
2. Argue that the transition amplitude for the $\psi\psi \to \psi\psi$ process reads

   $$(2\pi)^4\delta^{(4)}(p_3+p_4-p_1-p_2)i\mathcal{M} = -\frac{g^2}{2}\int \mathrm{d}^4x\int \mathrm{d}^4x'\,\langle f|:\bar{\psi}(x)\psi(x)\bar{\psi}(x')\psi(x'):|i\rangle\,\mathcal{D}_F(x,x') \ ,$$

   with $\mathcal{D}_F(x,x')$ the Feynman propagator for the scalar field.
3. Evaluate the above and find the amplitude $\mathcal{M}$.
   Hint : You should get two different contributions.
4. Compute

   $$|\bar{\mathcal{M}}|^2 = \frac{1}{4}\sum_{spins}|\mathcal{M}|^2 \ ,$$

   where the factor of $\frac{1}{4}$ accounts for the fact that there are 4 different initial spin configurations. Rewrite the above in terms of the Mandelstam variables.

5. Working in the center-of-mass frame, compute the differential cross-section for the process under consideration.
   Hint : Some relevant formulas may be found in the previous Problem Sets.

Part B

Take now the following Lagrangian

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi + \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}M^2\phi^2 - g\phi\bar{\psi}\gamma_5\psi$$

with $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$. Repeat Part B.

2. Maxwell Theory

Part A : Classical solutions of Maxwell equations

The Maxwell equations read

$$\partial^\mu F_{\mu\nu} = -j_\nu \ ,$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field-strength tensor and j_μ a conserved current.

1. Show that the Maxwell equations in momentum space read
$$P_{\mu\nu}A^\nu(k) = -j_\mu(k) \ , \tag{1}$$
   where the (transverse) wave operator $P_{\mu\nu}$ is
$$P_{\mu\nu} = -\eta_{\mu\nu}k^2 + k_\mu k_\nu \ , \quad \text{with} \quad k^2 = k_\mu k^\mu \ .$$
2. Write $P_{\mu\nu}$ as a 4×4 matrix and show that it is not invertible.
3. In order to solve equation (1), we need to restrict ourselves to specific gauges. Working in the Lorentz ($k_\mu A^\mu = 0$) and Coulomb ($k_i A^i = 0$) gauges, solve equation (1).

Part B : Mode decomposition of the electromagnetic field

In what follows we will be working in the Lorenz gauge $\partial_\mu A^\mu = 0$. The vector field can be decomposed as

$$A^\mu(x) = \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3 2\omega_{\vec{k}}} \sum_{r=0}^{3} \left(\epsilon_r^\mu(\vec{k}) a_r(\vec{k}) e^{-ikx} + \epsilon_r^\mu(\vec{k}) a_r^*(\vec{k}) e^{ikx} \right) \ ,$$

where the a's are (complex) coefficients and ϵ's are four polarization vectors.

1. For simplicity, let's work in the reference frame where $k^\mu = \omega_{\vec{k}}(1, 0, 0, 1)$. In this case, we can choose the polarization vectors to be
$$\epsilon_0^\mu = (1, 0, 0, 0) \ , \quad \epsilon_1^\mu = (0, 1, 0, 0) \ , \quad \epsilon_2^\mu = (0, 0, 1, 0) \ , \quad \epsilon_3^\mu = (0, 0, 0, 1) \ .$$
   Show that $\partial_\mu A^\mu = 0$ corresponds to $a_0(\vec{k}) = a_3(\vec{k})$.

2. Show that the polarization vectors satisfy

$$\eta_{\mu\nu}\epsilon_r^\mu\epsilon_s^\nu = -\zeta_s\delta_{rs}, \quad \sum_{r=0}^{3}\zeta_r\epsilon_r^\mu\epsilon_r^\nu = -\eta^{\mu\nu},$$

with $\zeta_0 = -1$ and $\zeta_{1,2,3} = 1$. Note that the above properties hold also for a general set of polarization vectors.

3. Compute the energy-momentum tensor of the Maxwell theory. Write the Hamiltonian in terms of a's.

4. Using again the above polarization vectors, discuss which polarizations contribute to the Hamiltonian.